WAVE INVARIANTS FOR NON-DEGENERATE CLOSED GEODESICS

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0. Introduction

This paper is a continuation of [Z.1]. There, we gave an effective method for putting the Laplacian Δ of a Riemannian manifold (M, g) into a quantum Birkhoff normal form around a non-degenerate elliptic closed geodesic γ, and applied it to the calculation and characterization of the wave invariants $a_{\gamma k}$ at γ. The wave invariants, we recall, are the coefficients in the singularity expansion

$$TrU(t) = e_o(t) + \sum_\gamma e_\gamma(t)$$
$$e_\gamma(t) \sim a_{\gamma\ -1}(t - L_\gamma + i0)^{-1} + \sum_{k=0}^\infty a_{\gamma\ k}(t - L_\gamma + i0)^k log(t - L_\gamma + i0)$$

of the trace of the wave group $U(t) = e^{it\sqrt{\Delta}}$ at lengths $t = L_\gamma$ of closed geodesics. The first purpose of this article is to extend the methods and results of [Z.1] to general non-degenerate closed geodesics, i.e. to γ whose Poincare map P_γ is any symplectic sum of non-degenerate elliptic, hyperbolic, or loxodromic parts. Our second purpose is to generalize to the full non-degenerate case the inverse result of Guillemin that the quantum normal form coefficients at non-degenerate elliptic closed geodesics are spectral invariants [G.1,2]. It will follow that, for metrics with simple length spectra, the classical Birkhoff normal form of the metric around any non-degenerate closed geodesic is a spectral invariant of the Laplacian.

Let us state the results precisely. The first is that the wave invariants $a_{\gamma k}$ in the general non-degenerate case are essentially analytic continuations of the expressions obtained in the elliptic case. They may be written in the form

$$a_{\gamma k} = \mathcal{F}_{k,-1}(D) \cdot Ch(x)|_{x=P_\gamma} \tag{0.1}$$

where

$$Ch(x) = \frac{i^\sigma}{\sqrt{|det(I - x)|}}$$

is the character of the metaplectic representation (with σ a certain Maslov index) and where $\mathcal{F}_{k,-1}(D)$ is an invariant partial differential operator on the metaplectic group $Mp(2n, \mathbb{R})$ which is canonically fashioned from the germ of the metric g at γ. The exact expression for $\mathcal{F}_{k,-1}(D)$ will be given in §5 and leads to the following characterization of the wave invariants (cf. [Z.1, Theorem A]):

Theorem I *Let γ be a non-degenerate closed geodesic. Then $a_{\gamma k} = \int_\gamma I_{\gamma;k}(s; g)ds$ where:*

(i) $I_{\gamma;k}(s, g)$ is a homogeneous Fermi-Jacobi-Floquet polynomial of weight -k-1 in the data $\{y_{ij}, \dot{y}_{ij}, D^\beta_{s,y} g\}$ with $|\beta| \le 2k + 4$;

(ii) The degree of $I_{\gamma;k}$ in the Jacobi field components is at most 6k+6;

(iii) At most 2k+1 indefinite integrations over γ occur in $I_{\gamma kr}$;

(iv) The degree of $I_{\gamma;k}$ in the Floquet invariants β_j is at most k+2.

The relevant terminology and notation will be recollected in §1. From (0.1) and from the formulae for $\mathcal{F}_{k,-1}(D)$ and Ch, we give a rather simple proof (§6) of the following inverse result (strictly speaking, proven only for non-degenerate elliptic closed geodesics in [G.2, Theorem 1.4]):

Theorem II *Let γ be a non-degenerate closed geodesic. Then the entire quantum Birkhoff normal form around γ is a spectral invariant; in particular the classical Birhoff normal form is a spectral invariant.*

We thus have:

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Corollary II.1 *Suppose (M, g) is a compact Riemannian manifold with simple length spectrum $Lsp(M, g)$ and with all closed geodesics non-degenerate. Then from $Spec(M, g)$ one can recover the quantum (and hence classical) Birkhoff normal forms around all closed geodesics.*

The hypotheses of the corollary are of course satisfied by generic Riemannian metrics (cf. [Kl, Lemma 4.4.3]). The corollary therefore answers affirmatively the third question in [Z.2, p.692], which asks whether isospectral manifolds in this class of metrics are locally Fourier isospectral near corresponding closed geodesics.

Let us now briefly discuss the main ideas in the proofs, and in particular the novel aspects caused by the hyperbolic and loxodromic parts of P_γ.

As in [G.1,2][Z.1], the wave invariants at a closed geodesic γ will be expressed as non-commutative residues of the wave group and its time derivatives at $t = L_\gamma$. For simplicity we will often abbreviate L_γ by L. Then we have:

$$a_{\gamma k} = res\ D_t^k e^{it\sqrt{\Delta}}|_{t=L} := Res_{s=0} Tr D_t^k e^{it\sqrt{\Delta}} \sqrt{\Delta}^{-s}|_{t=L}.$$

Since res is invariant under conjugation by (microlocal) unitary Fourier integral operators, the $a_{\gamma k}$'s may be calculated by putting the wave group into a microlocal (quantum Birkhoff) normal form around γ and by by determining the residues of the resulting wave group of the normal form.

The primary step in the analysis of the wave invariants $a_{\gamma k}$ is therefore to put Δ into this microlocal normal form around γ. In the case of non-degenerate elliptic closed geodesics, we recall, the normal form was a polyhomogeneous function in the (microlocally elliptic) element D_s with coefficients in the transverse (elliptic) harmonic oscillators $\hat{I}_j^e = \frac{1}{2}(D_{y_j}^2 + y_j^2)$ [Z.1,Theorem B]. In the general non-degenerate case, the normal form will involve a greater variety of quadratic normal forms or 'action operators:' in addition to the elliptic action operator $\hat{I}_j^e$ there can also occur the real hyperbolic action operators $\hat{I}_j^h$ and complex hyperbolic (or loxodromic) action operators $\hat{I}_j^{ch,Re}, \hat{I}_j^{ch,Im}$. These hyperbolic actions cause several complications to the arguments in the elliptic case: First, they have continuous spectra, and so the construction of the intertwining operator to the normal form has to be modified in several ways (§3,4). Second, the wave group of the normal form has continuous spectrum and this alters the calculation of its residues (§5). Third, the presence of real parts in the Floquet exponents of P_γ complicates the process of determining the normal form coefficients from the wave invariants (§6).

To get acquainted with these action operators and the normal form algorithm, let us consider the very first step of "linearizing" $\sqrt{\Delta}$ around γ and of putting the 'linearization'

$$\mathcal{L} = D_s - \frac{1}{2}(\sum_{j=1}^n D_{y_j}^2 + \sum_{ij=1}^n K_{ij} y_i y_j)$$

into quantum quadratic normal form. Here, $n = dimM - 1$, the coordinates (s, y_j) are the (re-scaled) Fermi normal coordinates around γ, $D_s = \frac{\partial}{i\partial s}$ and K_{ij} is the curvature operator $g(R(\partial_s, \partial_{y_i})\partial_s, \partial_{y_j})$. The linearization $\mathcal{L}$ of $\sqrt{\Delta}$ is a quadratic Hamiltonian and is the Weyl quantization of a quadratic classical Hamiltonian (see [Ho III] and [Ho] for background on Weyl quantizations and normal forms for quadratic Hamiltonians). Hence its symbol may be conjugated into normal form by an element of $\mathcal{W} \in Sp(2n, \mathbb{R})$. The operator $\mathcal{L}$ itself may be put into normal form by conjugating with the metaplectic operator $\mu(\mathcal{W})$ quantizing $\mathcal{W}$. As will be seen in §1 , this linear symplectic map is the Wronskian matrix $\mathcal{W}$ whose columns consist of the Jacobi eigenfields of P_γ. The normal form of the linearized $\sqrt{\Delta}$ is therefore given by $\mathcal{R} = \mu(\mathcal{W})^* \mathcal{L} \mu(\mathcal{W})^{*-1}$.

In the elliptic case [Z.1] P_γ was a direct sum of rotations, and the quantum normal form of $\mathcal{L}$ had the form

$$\mathcal{R}^e = D_s + \frac{1}{L} H_\alpha, \qquad H_\alpha = \sum_{j=1}^n \alpha_j \hat{I}_j^e$$

where the spectrum $\sigma(P_\gamma) = \{e^{\pm i\alpha_j}\}$. In the general non-degenerate case the normal form will similarly depend on the spectral decomposition of P_γ. Recall that, since P_γ is symplectic, its eigenvalues ρ_j come in three types: (i) pairs $\rho, \bar{\rho}$ of conjugate eigenvalues of modulus 1; (ii) pairs ρ, ρ^{-1} of inverse real eigenvalues; and (iii) 4-tuplets $\rho, \bar{\rho}, \rho^{-1}\bar{\rho}^{-1}$ of complex eigenvalues. We will often write them in the forms: (i) $e^{\pm i\alpha_j}$, (ii)$e^{\pm\lambda_j}$, (iii) $e^{\pm\mu_j \pm i\nu_j}$ respectively (with $\alpha_j, \lambda_j, \mu_j, \nu_j \in \mathbb{R}$), although a pair of inverse real eigenvalues

$\{-e^{\pm\lambda}\}$ could be negative. Here, and throughout, we make the assumption that P_γ is *non-degenerate* in the sense that

$$\Pi_{i=1}^{2n}\rho_i^{m_i} \neq 1, \qquad (\forall \rho_i \in \sigma(P_\gamma), \quad (m_1,\dots,m_{2n}) \in \mathbf{N}^{2n}).$$

Each type of eigenvalue then determines a different type of quadratic action, both on the classical and quantum levels (cf. [Ho, Theorem 3.1],[Ar]):

Eigenvalue type	Classical Normal form	Quantum normal form
(i) Elliptic type $\{e^{i\pm\alpha}\}$	$I^e = \frac{1}{2}\alpha(\eta^2+y^2)$	$\hat{I}^e := \frac{1}{2}\alpha(D_y^2+y^2)$
(ii) Real hyperbolic type $\{e^{\pm\lambda}\}$	$I^h = 2\lambda y\eta$	$\hat{I}^h := \lambda(yD_y + D_y y)$
(iii) Complex hyperbolic (or loxodromic type) $\{e^{\pm\mu+i\pm\nu}\}$	$I^{ch,Re} = 2\mu(y_1\eta_1 + y_2\eta_2)$ $I^{ch,Im} = \nu(y_1\eta_2 - y_2\eta_1)$	$\hat{I}^{ch,Re} = \mu(y_1D_{y_1} + D_{y_1}y_1 + y_2D_{y_2} + D_{y_2}y_2)$, $\hat{I}^{ch,Im} = \nu(y_1D_{y_2} - y_2D_{y_1})$

In the case where the Poincare map P_γ has p pairs of complex conjugate eigenvalues of moduls 1, q pairs of inverse real eigenvalues and c quadruplets of complex hyperbolic eigenvalues, the linearized $\sqrt{\Delta}$ will have the form:

$$\mathcal{R} = D_s + \frac{1}{L}[\sum_{j=1}^{p}\alpha_j\hat{I}_j^e + \sum_{j=1}^{q}\lambda_j\hat{I}_j^h + \sum_{j=1}^{c}\mu_j\hat{I}_j^{ch,Re} + \nu_j\hat{I}_j^{ch,Im}].$$

The full quantum Birkhoff normal form is then given by the analogue of Theorem B of [Z.1]:

Theorem B *Assuming γ non-degenerate, there exists a microlocally elliptic Fourier integral operator W from the conic neighborhood of $\mathbb{R}^+\gamma$ in $T^*(N_\gamma)$ to the corresponding cone in $T^*_+S^1$ in $T^*(S^1\times\mathbb{R}^n)$ such that*

$$W\sqrt{\Delta}W^{-1} \equiv D_s + \frac{1}{L}[\sum_{j=1}^{p}\alpha_j\hat{I}_j^e + \sum_{j=1}^{q}\lambda_j\hat{I}_j^h + \sum_{j=1}^{c}\mu_j\hat{I}_j^{ch,Re} + \nu_j\hat{I}_j^{ch,Im}] +$$
$$+\frac{p_1(\hat{I}_1^e,\dots,\hat{I}_p^e,\hat{I}_1^h,\dots,\hat{I}_q^h,\hat{I}_1^{ch,Re},\hat{I}_1^{ch,Im},\dots,\hat{I}_c^{ch,Re},\hat{I}_c^{ch,Im})}{D_s} + \dots$$
$$+\frac{p_{k+1}(\hat{I}_1^e,\dots,\hat{I}_c^{ch,Im})}{D_s^k} + \dots$$

where the numerators $p_j(\hat{I}_1^e,\dots,\hat{I}_p^e,\hat{I}_1^h,\dots,\hat{I}_c^{ch,Im})$ are polynomials of degree j+1 in the variables $(\hat{I}_1^e,\dots,\hat{I}_c^{ch,Im})$ and where the kth remainder term lies in the space $\oplus_{j=o}^{k+2}O_{2(k+2-j)}\Psi^{1-j}$

Here, $O_n\Psi^r$ is the space of pseudodifferential operators of order r whose complete symbols vanish to order n at $(y,\eta) = (0,0)$. Thus, the remainder terms are 'small' in that they combine in some mixture a low pseudodifferential order or a high vanishing order along γ.

Some remarks now on the contents and organization of this paper. Since it is a continuation of [Z.1], we have tried to avoid duplicating arguments and calculations which are essentially unchanged from the elliptic case. Many of the arguments which remain are still quite analogous to the elliptic case and inevitably produce a sense of deja-vu. Our excuse for drawing them out to their present length is that it is not apriori clear that the arguments of the elliptic case generalize so neatly to the hyperbolic and loxodromic cases. It may even be viewed as a virtue of the method of [Z.1] that it adapts so effortlessly to the general case.

It should be noted here that Guillemin was aware that the arguments of the elliptic case should extend to the general non-degenerate case and stated his main result, Theorem 1.4 of [G.2], for the general case. However, we also note that the methods used here are extensions of the methods of [Z.1], which in many significant respects differ from the methods of [G.2].

The organization of this paper is as follows: In §1 we will review the symplectic and microlocal ingredients required to construct a 'linear model' for the Laplacian near a closed geodesic γ. In §2, we introduce the semi-classically scaled Laplacian and the linearized Laplacian and conjugate them to the model space. In §3, we conjugate the resulting semi-classical model Laplacian to a semi-classical normal form to infinite order. In §4 we show how this semi-classical normal form induces a bona-fide quantum Birkhoff normal

form for the Laplacian near γ. In §5, we use the normal form to give the explicit formula (0.1) for the wave invariants. In §6, we show that the quantum normal form coefficients can be determined from the special values of (0.1) corresponding to γ and its iterates.

1. Preliminaries

This section begins with a resume of the symplectic linear algebra underlying the Jacobi equation, the linear Poincare map, and the symplectic classification of non-degenerate quadratic forms (§1.1). It then summarizes the quantum aspects of the linear theory, in particular the behaviour of the quantum action operators (§1.2)

§1.1: Symplectic preliminaries

§1.1a: Closed geodesics, linear Poincare maps and Jacobi fields

Throughout this paper, γ will denote a *primitive* closed geodesic of (M, g); its iterates will be denoted by γ^m.

The space $\mathcal{J}_\gamma^\perp$ of (real) orthogonal Jacobi fields along γ is then the real symplectic vector space, of dimension 2n, of solutions of the Jacobi equation $Y'' + R(T, Y)T = 0$ (with T the unit tangent vector along γ.) The symplectic structure is given by the Wronskian

$$\omega(X, Y) = g(X, \frac{D}{ds}Y) - g(\frac{D}{ds}X, Y).$$

The linear Poincare map P_γ is the (real) linear symplectic map on $(\mathcal{J}_\gamma^\perp, \omega)$ defined by $P_\gamma Y(t) = Y(t + L_\gamma)$. To diagonalize it, we also complexify it as a complex symplectic map $P_\gamma^{\mathbb{C}}$ on the space $\mathcal{J}_\gamma^\perp \otimes \mathbb{C}$ of complex orthogonal Jacobi fields. Here, the symplectic form is extended to the complexified space as a complex bilinear form $\omega^{\mathbb{C}}$. Since $P_\gamma^{\mathbb{C}} \in Sp(\mathcal{J}_\gamma^\perp \otimes \mathbb{C}, \omega)$, its spectrum $\sigma(P_\gamma^{\mathbb{C}})$ is stable under inverse and complex conjugation: thus, if $\rho \in \sigma(P_\gamma^{\mathbb{C}})$, then also $\rho^{-1}, \bar{\rho}, \bar{\rho}^{-1} \in \sigma(P_\gamma^{\mathbb{C}})$. As mentioned above, we will assume that $P_\gamma^{\mathbb{C}}$ is *non-degenerate* in the following strong sense:

$$(1.1a.1) \qquad \rho_1^{m_1} \dots \rho_n^{m_n} = 1 \Rightarrow m_i = 0 \quad (\forall i, m_i \in \mathbb{N}).$$

In particular, the eigenvalues are simple and $\pm 1 \notin \sigma(P_\gamma^{\mathbb{C}})$.

The eigenspace of $P_\gamma^{\mathbb{C}}$ of eigenvalue ρ will be denoted by $\mathcal{J}_\gamma^{\perp,\mathbb{C}}(\rho) \subset \mathcal{J}_\gamma^\perp \otimes \mathbb{C}$. We then have the symplectic orthogonal decomposition

$$(1.1a.2) \qquad \mathcal{J}_\gamma^{\perp,\mathbb{C}} \otimes \mathbb{C} = \mathcal{J}_{nc}^{\perp,\mathbb{C}} \oplus \mathcal{J}_{co}^{\perp,\mathbb{C}}$$

into the 'non-compact' symplectic subspace

$$(1.1a.3.nc) \qquad \mathcal{J}_{nc}^{\perp,\mathbb{C}} = \oplus_{\rho:|\rho|\neq 1} \mathcal{J}_\gamma^{\perp,\mathbb{C}}(\rho)$$

where $P_\gamma^{\mathbb{C}}$ does not belong to a compact subgroup of Sp and the 'compact' symplectic subspace

$$(1.1a.3.co) \qquad \mathcal{J}_{co}^{\perp,\mathbb{C}} = \oplus_{\rho:|\rho|=1} \mathcal{J}_\gamma^{\perp,\mathbb{C}}(\rho)$$

where $P_\gamma^{\mathbb{C}}$ does belong to a compact subgroup of Sp. This and the following decompositions are described in more detail in Klingenberg [Kl], but also somewhat differently since in [Kl] the symplectic form is extended to the complexification as a sesquilinear form rather than as a complex bilinear form.

The non-compact subspace has the further symplectic orthogonal decompositon

$$(1.1a.4) \qquad \mathcal{J}_{nc}^{\perp,\mathbb{C}} = \bigoplus_{\rho:|\rho|<1} [\mathcal{J}_\gamma^{\perp,\mathbb{C}}(\rho) \oplus \mathcal{J}_\gamma^{\perp,\mathbb{C}}(\rho^{-1})]$$

into symplectic complex 2-planes. We may rewrite this decomposition in the form

$$(1.1a.5) \qquad \mathcal{J}_{nc}^{\perp,\mathbb{C}} = \mathcal{J}_s^{\perp,\mathbb{C}} \oplus \mathcal{J}_u^{\perp,\mathbb{C}}$$

where

$$\mathcal{J}_s^{\perp,\mathbb{C}} = \bigoplus_{\rho:|\rho|<1} \mathcal{J}_\gamma^{\perp,\mathbb{C}}(\rho), \qquad \mathcal{J}_u^{\perp,\mathbb{C}} = \bigoplus_{\rho:|\rho|>1} \mathcal{J}_\gamma^{\perp,\mathbb{C}}$$

are the symplectically dual stable, resp. unstable Lagrangean subspaces.

The compact subspace has the further symplectic decompositon

$$\mathcal{J}_{co}^{\perp,\mathbb{C}} = \bigoplus_{\rho:\rho=e^{i\alpha},\alpha\in(0,\pi)} \mathcal{J}_{\gamma}^{\perp,\mathbb{C}}(\rho) \oplus \mathcal{J}_{\gamma}^{\perp,\mathbb{C}}(\bar{\rho}). \tag{1.1a.6}$$

Any choice of one ρ from a pair $\{\rho,\bar{\rho}\}$ determines a splitting of $\mathcal{J}_{co}^{\perp,\mathbb{C}}$ into a pair of dual Lagrangean subspaces.

On the level of real symplectic spaces, we have the closely related P_γ-invariant symplectic decomposition

$$\mathcal{J}_{\gamma}^{\perp} = \mathcal{J}_{s}^{r} \oplus \mathcal{J}_{u}^{r} \oplus \mathcal{J}_{ce}^{\perp,2p} \tag{1.1a.7}$$

into the stable, unstable and center stable real subspaces of dimensions $r, r, 2p$ respectively. By definition,

$$\mathcal{J}_{s}^{r} = \bigoplus_{\rho\in\mathbb{R},|\rho|<1} \mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\rho) \oplus \bigoplus_{\rho\in\mathbb{C}-\mathbb{R},|\rho|<1} \mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\rho) \tag{1.1a.8}$$

where

$$P_\gamma|_{\mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\rho)} = \rho \qquad (\rho\in\mathbb{R})$$

respectively

$$P_\gamma|_{\mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\rho)} = e^{-\mu}\begin{pmatrix} cos\nu & sin\nu \\ -sin\nu & cos\nu \end{pmatrix} \qquad \rho = e^{-\mu+i\nu}, \ \ \mu,\nu>0.$$

In the latter case, $\mathcal{J}_{\gamma}^{\perp,\mathbb{C}-\mathbb{R}}(\rho)$ is the real symplectic 2-plane whose complexification equals $\mathcal{J}_{\gamma}^{\perp,\mathbb{C}}(\rho) \oplus \mathcal{J}_{\gamma}^{\perp,\mathbb{C}}(\rho^{-1})$. Similarly for the case of the unstable subspace. In the center stable case,

$$\mathcal{J}_{\gamma,ce}^{\perp,2p} = \bigoplus_{j=1}^{p} \mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\alpha_j) \tag{1.1a.9}$$

where

$$P_\gamma|_{\mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\alpha)} = \begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix}$$

and with $\mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\alpha)$ the symplectic two plane whose complexification equals $\mathcal{J}_{\gamma}^{\perp,\mathbb{C}}(\rho) \oplus \mathcal{J}_{\gamma}^{\perp,\mathbb{C}}(\bar{\rho})$ with $\rho = e^{i\alpha}$.

We will put $r = q + 2c$ where $q = \#\{\rho\in\mathbb{R}, |\rho|<1\}$ and where $2c = \#\{\rho\in\mathbb{C}-\mathbb{R}, |\rho|<1\}$ and say that γ has type (p,q,c) if it has p pairs of conjugate eigenvalues of modulus one $\{e^{i\alpha}, e^{-i\alpha}\}$, q pairs of real inverse eigenvalues $\{e^{\lambda}, e^{-\lambda}\}$, and c quadruples of non-real complex eigenvalues $\{e^{\pm\mu+\pm i\nu}\}$. Here, we have assumed the real eigenvalues are positive for brevity of notation. Finally we may write:

$$\mathcal{J}_{\gamma}^{\perp} = \mathcal{J}_{\gamma}^{e} \oplus \mathcal{J}_{\gamma}^{h} \oplus \mathcal{J}_{\gamma}^{ch} \tag{1.1a.10}$$

where $\mathcal{J}_{\gamma}^{e} = \mathcal{J}_{\gamma,ce}^{\perp,2p}$ is the *elliptic* (or real center stable) subspace, where

$$\mathcal{J}_{\gamma}^{h} = \bigoplus_{\rho\in\mathbb{R},|\rho|<1} \mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\rho) \oplus \mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\rho^{-1}) \tag{1.1a.11}$$

is the *real hyperbolic* subspace and where

$$\mathcal{J}_{\gamma}^{ch} = \bigoplus_{\rho\in\mathbb{C}-\mathbb{R},|\rho|<1} \mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\rho) \oplus \mathcal{J}_{\gamma}^{\perp,\mathbb{R}}(\rho^{-1}) \tag{1.1a12}$$

is the *complex hyperbolic* (or loxodromic) subspace.

§1.1b: Jacobi eigenvectors and Wronskian matrix

As mentioned in the introduction, the intertwining operator to the quantum normal form will involve a certain Wronskian matrix of the Jacobi equation. Roughly speaking, it is the real symplectic matrix whose entries are given by the real and imaginary parts of the Jacobi eigenvectors and their time derivatives relative to a normal frame.

More precisely, we fix a symplectic orthonormal basis of Jacobi eigenvectors as follows:

complex subspace	eigenvectors	normalization
elliptic plane	$P_\gamma Y_j^e = e^{i\alpha_j} Y_j^e,\ P_\gamma Y^e{}_j = e^{-i\alpha_j} Y^e{}_j$	$\omega(Y_j^e, Y_j^e) = 1.$
real hyp. plane	$P_\gamma Y_j^+ = e^{\lambda_j} Y_j^+,\ P_\gamma Y_j^- = e^{-\lambda_j} Y_j^-$	$\omega(Y_j^+, Y_j^-) = 1.$
cx.hyp.4-plane	$P_\gamma Y_j^{++} = e^{\mu+i\nu} Y_j^{++},\ P_\gamma Y_j^{--} = e^{-\mu-i\nu} Y_j^{--},$	$\omega(Y^{++}, Y^{--}) = 1$
	$P_\gamma Y_j^{+-} = e^{\mu-i\nu} Y_j^{+-},\ P_\gamma Y_j^{-+} = e^{-\mu+i\nu} Y_j^{-+},$	$\omega(Y^{-+}, Y^{+-}) = 1$

The normalization makes sense since complex 2-planes spanned by eigenvectors corresponding to inverse eigenvalues are symplectic.

We now fix a parallel normal frame $e(s) := (e_1(s), ..., e_n(s))$ along $\gamma|_{[O,L)}$ and denote by $\langle Y, e_j\rangle$ the Riemannian inner product of a vector Y along γ with the the jth normal vector. Corresponding to the splitting of $\mathcal{J}_\gamma^\perp$ into its elliptic, hyperbolic and and complex hyperbolic (real) subspaces, we then get a real symplectic 2n x 2n Wronskian matrix

$$\mathcal{W} = [\mathcal{W}^e|\mathcal{W}^h|\mathcal{W}^{ch}] \tag{1.1b.1}$$

formed by the 2n x 2p elliptic Wronskian matrix

$$\mathcal{W}^e(s) := \begin{pmatrix} Re\langle Y_i^e, e_j\rangle & Im\langle Y_i^e, e_j\rangle \\ Re\langle \dot{Y}_i^e, e_j\rangle & Im\langle \dot{Y}_i^e, e_j\rangle \end{pmatrix}_{i=1,\dots,p; j=1,\dots,n} \tag{1.1b.2e}$$

the 2n x 2q real hyperbolic Wronskian matrix

$$\mathcal{W}^h(s) := \begin{pmatrix} \langle Y_i^+, e_j\rangle & \langle Y_i^-, e_j\rangle \\ \langle \dot{Y}_i^+, e_j\rangle & \langle \dot{Y}_i^-, e_j\rangle \end{pmatrix}_{i=1,\dots,q; j=1,\dots,n} \tag{1.1b.2h}$$

and finally the complex hyperbolic 2n x 4c Wronskian matrix

$$\mathcal{W}^{ch}(s) := \begin{pmatrix} Re\langle Y_i^{++}, e_j\rangle & Im\langle Y_i^{++}, e_j\rangle & Re\langle Y_i^{--}, e_j\rangle & Im\langle Y_i^{--}, e_j\rangle \\ Re\langle \dot{Y}_i^{++}, e_j\rangle & Im\langle \dot{Y}_i^{++}, e_j\rangle & Re\langle \dot{Y}_i^{--}, e_j\rangle & Im\langle \dot{Y}_i^{--}, e_j\rangle \end{pmatrix}_{i=1,\dots,c; j=1,\dots,n}. \tag{1.1b.2ch}$$

To see that $\mathcal{W}$ is indeed symplectic, and to better understand its properties, we reconsider the Jacobi equation and Poincare map from the Riemannian viewpoint. Thus, we let ∇ denote the Riemannian connection, and recall that it determines a horizontal subbundle of $T(S^*M)$ complementary to the vertical subbundle of the projection $\pi : S^*M \to M$. Together with the symplectic structure, we get a splitting

$$T(S^*M) = \bar{H} \oplus \bar{V} \oplus \bar{T}$$

where $\bar{T}$ is the real span of $\dot{\gamma}$, and $\bar{H} \oplus \bar{V}$ is the horizontal plus vertical decomposition of the kernel of the contact form $\alpha = \xi \cdot dx$ (or equivalently, of the symplectic orthogonal of T and the cone axis). The subspaces $\bar{H}, \bar{V}$ are symplectically paired Lagrangean subspaces of $T(T^*M)$. Given a vector $X \in N_{\gamma(t)}$, we denote by X^h the horizontal lift of X to $\bar{H}_{\gamma(t)}$ and by X^v the vertical lift to $\bar{V}_{\gamma(t)}$. The correspondence

$$Y(t) \to (Y(t)^h, \dot{Y}(t)^v)$$

then defines an isomorphism between the spaces of Jacobi fields and geodesic flow invariant vector fields along $(\gamma(t), \dot{\gamma}(t)$ (cf [Kl, Lemma 3.1.6]). That is,

$$dG^s_{(\gamma(0),\dot{\gamma}(0))} : (Y(0)^h, \dot{Y}(0)^v) \to (Y(s)^h, \dot{Y}(s)^v)$$

where $Y(s)$ is the Jacobi field with the given initial conditions. Moreover, since G^t is a Hamiltonian flow dG^s is a linear symplectic mapping from $(\bar{H} \oplus \bar{V})_{\gamma(0),\dot{\gamma}(0)}$ to $(\bar{H} \oplus \bar{V})_{\gamma(s),\dot{\gamma}(s)}$.

Relative to the basis $\{e_j(s)\}$, the Jacobi equation is equivalent to the linear system $\frac{D}{ds}(Y, P) = JH(Y, P)$, where: (i) $P = \frac{DY}{ds}$, (ii) J is the standard complex structure on R^{2n}, and where (iii)

$$H = \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix}$$

with K the curvature matrix and with I the identity matrix. Moreover the basis $\{e_j(s)\}$ induces a moving symplectic frame $\{e_j^h(s), e_j^v(s)\}$ of $(\bar{H} \oplus \bar{V})_{\gamma(s),\dot{\gamma}(s)}$. The evolution operator for the linear system is then just dG^s expressed as a matrix relative to the moving symplectic frame.

Now consider the above basis $\{ReY_j^e, ImY_j^e, Y_j^+, Y_j^-, ReY_j^{++}, ImY_j^{++}, ReY_j^{--}, ImY_j^{--}\}$ of $\mathcal{J}_\gamma^\perp$ formed by the eigenvectors of P_γ. By construction, it is a symplectic basis relative to the Wronskian form ω. Hence the pairs $(Y^h(s), P^v(s))$ consisting of these eigenvectors and their time derivatives form a moving symplectic basis of $\bar{H} \oplus \bar{V}$. Expressed in terms of the frame $\{e_j^h(s), e_j^v(s)\}$ we then get a symplectic basis of $\mathbb{R}^{2n}$ relative to the standard symplectic structure. The Wronskian matrix $\mathcal{W}(s)$ is just the matrix whose columns are formed by these basis elements, and it is therefore symplectic for each s.

Consider now the monodromy aspect of $\mathcal{W}(s)$, i.e. its transformation law under time translation $s \to s + L$ thru one period. Let Y_i denote one of the complex eigenvectors of P_γ. Then the matrix element $\langle Y_i(s), e_j(s)\rangle$ (or with $\dot{Y}_i$ in place of Y_i) satisfies

$$\langle Y_i(s+L), e_j(s+L)\rangle = \rho_i \sum_{k=1}^n t_{jk} \langle Y_i(s), e_k(s)\rangle \tag{1.1b.3}$$

where $T := (t_{jk})$ is the holonomy matrix,

$$e_j(s+L) = \sum_{k=1}^n t_{jk} e_k(s).$$

It follows that

$$\mathcal{W}(s+L) = \mathcal{W}(s) T^* P_\gamma. \tag{1.1b.4}$$

§1.1c: Symplectic equivalence of quadratic Hamiltonians

Let $(\mathbb{R}^{2n}, \omega)$ be the standard symplectic vector space, endowed with linear coordinates $x = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ such that $\omega = \sum dq_i \wedge dp_i$. A quadratic (real) Hamiltonian is by definition a quadratic form

$$H(q,p) = \frac{1}{2}\langle Ax, x\rangle = \frac{1}{2}\omega(JAx, x)$$

where $\langle \cdot \rangle$ is the Euclidean scalar product, where A is a 2n x 2n real symmetric matrix and where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Then $JA \in sp(\mathbb{R}^{2n}, \omega)$, and hence its spectrum decomposes into purely imaginary pairs $(i\alpha, -i\alpha)$, into real pairs $(\lambda, -\lambda)$, and into complex quadruples $(\pm\mu \pm i\nu)$. We will assume, as above, that the eigenvalues are simple and not equal to ± 1. Then $H(q,p)$ decomposes into sums of terms of the following types of quadratic Hamiltonians, or classical 'actions': the elliptic type

$$I^e(q_1, p_1) := \frac{1}{2}\alpha(q_1^2 + p_1^2), \tag{1.1c.1e}$$

the real hyperbolic type

$$I^h(q_1, p_1) := \frac{1}{2}\lambda q_1 p_1, \tag{1.1c.1h}$$

and the complex hyperbolic (or loxodromic) type

$$I^{ch}(q_1, p_1, q_2, p_2) = \frac{1}{2}\mu(q_1p_1 + q_2p_2) + \frac{1}{2}\nu(q_1p_2 - q_2p_1). \tag{1.1c.1ch}$$

Note that

$$q_1p_1 + q_2p_2 = Re(q_1 + iq_2)(p_1 - ip_2), \qquad q_2p_1 - q_1p_2 = Im(q_1 + iq_2)(p_1 - ip_2)$$

and

$$\{q_1p_1 + q_2p_2, q_1p_2 - q_2p_1\} = 0$$

To unify these expressions, we observe that they all have the form $\frac{1}{2}Re\ sa^*a$ where $s \in \mathbb{C}$ and where a^*, a denote symplectically dual complex linear coordinates. Indeed, in the elliptic case, $a = q_1 + ip_1, a^* = q_1 - ip_1, s = \alpha \in \mathbb{R}$; in the real hyperbolic case, $a = q_1, a^* = p_1, s = \lambda \in \mathbb{R}$; and in the loxodromic case, $s = \mu + i\nu, a = (q_1 + iq_2), a^* = (p_1 - ip_2)$.

§1.2: Microlocal preliminaries

As mentioned above, the wave invariants only involve the metric and Laplacian Δ in a tubular neighborhood of γ. In fact, as discussed in [G.1][Z.1] they only involve the microlocalization of Δ to the conic neighborhood

$$|y| \leq \epsilon, \qquad |\eta| < \epsilon\sigma \tag{1.2.1}$$

of $T^*S^1_L - 0$ in $T^*(S^1 \times \mathbb{R}^n)$. Here,$(s, \sigma, y, \eta)$ denote the symplectic Fermi coordinates and ψ denotes a smooth homogeneous cut-off function on $T^*(S^1_L \times \mathbb{R}^n) - 0$ which equals 1 in some conic neighborhood V of $T^*S^1_L - 0$ and vanishes identically off of some slightly larger conic neighborhood.

As in the case of elliptic closed geodesics, to put Δ into a microlocal (quantum Birkhoff) normal form around $\gamma \sim S^1$ is first of all to conjugate it to a distinguished maximal abelian subalgebra $\mathcal{A}_\gamma$ of the algebra $\Psi^*(S^1_L \times \mathbb{R}^n)$ of pseudo-differential operators on the model space $S^1_L \times \mathbb{R}^n$. This distinguished subalgebra will depend on the type of the geodesic γ. Roughly speaking, it will consist of the tangential operator $D_s := \frac{1}{i}\frac{\partial}{\partial s}$ together with an appropriate set of quantized quadratic normal forms or 'action operators' in the transversal directions.

§1.2.1: The model algebras

To specify this 'appropriate set' of action operators, we begin by recalling that the Schrodinger representation of the (complexified) Heisenberg algebra $\mathbf{h}_n \otimes \mathbb{C}$ on the transverse space $L^2(\mathbb{R}^n)$, is generated by the self-adjoint operators $Y_j =$ "multiplication by y_j" and by $D_j = \frac{\partial}{i\partial y_j}$. Equivalently it is generated by the creation/annihilation operators $Y_j + iD_j, Y_j - iD_j$. For our purposes, however, it will be more natural to choose a different set of generators depending on the (q, p, c) type of the closed geodesic.

Corresponding to the $2p$ dimensional elliptic symplectic subspace we will use as generators the above (elliptic) annihilation/creation operators

$$Z_j := Y_j + iD_{y_j} \qquad Z_j^\dagger = Y_j - iD_{y_j} \quad (j = 1, \dots, p) \tag{1.2.1.1e}$$

which satisfy the commutation relations

$$[Z_j, Z_k] = [Z_j^\dagger, Z_k^\dagger] = 0 \qquad [Z_j, Z_k^\dagger] = 2\delta_{ij} I.$$

We would like to use the real (resp. complex) hyperbolic analogues in the hyperbolic subspaces. To determine the analogues we note that Z_j, resp. $Z_j^\dagger$ are the Weyl quantizations of the symplectically dual (modulo a factor of 2) complex linear coordinates $z_j := y_j + i\eta_j$, resp. $z_j^\dagger := y_j - i\eta_j$. We use the 'dagger' notation rather than the adjoint notation Z_j^* to emphasize that the dual operators are symplectically dual; they are also adjoints of each other, but this property will not extend to the hyperbolic cases. Indeed, corresponding to the $2q$ dimensional real hyperbolic subspace, the natural generators are the hyperbolic annihilation/creation operators

$$Y_j, \qquad D_{y_j} \quad (j = p+1, \dots, q) \tag{1.2.1.1h}$$

which of course are also symplectically dual. And corresponding to the $4c$ dimensional complex hyperbolic subspace, we the natural generators are the complex hyperbolic annihilation/creation operators

(1.2.1ch)

$$W_j := Y_j + iY_{c+j}, \quad W_j^\dagger := D_{y_j} - iD_{y_{c+j}}, \quad \bar{W}_j := Y_j - iY_{c+j}, \quad \bar{W}_j{}^\dagger := D_{y_j} + iD_{y_{c+j}} \quad (j = p+q+1, \dots, c).$$

We note that they satisfy the commutation relations:

$$[W_j, W_j^\dagger] = -2, \qquad [\bar{W}_j, \bar{W}_j{}^\dagger] = -2$$

with all other brackets zero. We will not bother to renormalize the operators to be precisely dual.

The enveloping algebra of the Heisenberg algebra is then generated by all the above annihilation/creation operators,

$$\mathcal{E} := < Z_1, \dots Z_p, Z_1^\dagger, \dots, Z_p^\dagger, Y_1, ..., Y_q, D_{y_1}, ..., D_{y_q}, W_1, \bar{W}_1, \dots, W_c, \bar{W}_c, W_1^\dagger, \bar{W}_1^\dagger, \dots, W_c^\dagger, \bar{W}_c^\dagger > \tag{1.2.1.2}$$

and is of course the algebra of partial differential operators on $\mathbb{R}^n$ with polynomial coefficients. We will denote by $\mathcal{E}^n$ the subspace of polynomials of degree n in the generators. The microlocalization of this

algebra is the isotropic Weyl algebra $\mathcal{W}^*$ of pseudo-differential operators on $\mathbb{R}^n$, in which the generators are assigned the order $\frac{1}{2}$, so that

$$\mathcal{E}^n \subset \mathcal{W}^{n/2}$$

$$[\mathcal{E}^m, \mathcal{E}^n] \subset \mathcal{E}^{m+n-2}.$$

The symplectic algebra $\mathbf{sp}(n, \mathbb{C})$ is then represented in $\mathcal{E}^2$ by homogeneous quadratic polynomials in the generators, which have degree 1. In particular it contains the following elliptic, resp. hyperbolic, resp. complex hyperbolic (loxodromic) 'action' operators:
(1.2.1.3)

$$\hat{I}_j^e := Z_j^\dagger Z_j, \qquad \hat{I}_j^h = \frac{1}{2}(Y_j D_{y_j} + D_{y_j} Y_j), \qquad \hat{I}_j^{ch,Re} = \frac{1}{2} Re(W_j^\dagger W_j + (W_j^\dagger W_j)^*), \qquad \hat{I}_j^{ch,Im} = Im W_j^\dagger W_j.$$

The complex hyperbolic action operators can also be written in the form

$$(1.2.1.4) \qquad \hat{I}_j^{ch,Re} = \frac{1}{2}(Y_j D_{y_j} + D_{y_j} Y_j + Y_{j+c} D_{y_{j+c}} + D_{y_{j+c}} Y_{j+c}), \qquad \hat{I}_j^{ch,Im} = (Y_j D_{y_{j+c}} - Y_{j+c} D_{y_j})$$

where the coordinates are indexed so that the $dy_j \wedge dy_{j+c} \wedge d\eta_j \wedge d\eta_{j+c}$-planes are the P_γ-invariant complex hyperbolic 4-planes. It is then natural to introduce polar coordinates r_j, ϕ_j on the (y_j, y_{j+c})-plane so that the loxodromic actions operators simplify to

$$(1.2.1.5) \qquad I_j^{ch,Re} = \frac{1}{2}(r_j D_{r_j} + D_{r_j} r_j), \qquad I^{ch,Im} = D_\theta.$$

We now introduce the distinguished (p, q, c) maximal (transverse) abelian subalgebra of $\mathcal{W}$, given by

$$(1.2.1.6) \qquad \mathcal{I}_{p,q,c} :=< I_1^e, ..., I_p^e, I_1^h, \dots, I_q^h, I_1^{ch,Re}, \dots, I_c^{ch,Re}, I_1^{ch,Im}, \dots, I_c^{ch,Im} > .$$

Together with the tangential operator we get the *(p,q,c)- maximal abelian subalgebra* given by

$$(1.2.1.7) \qquad \mathcal{A}_{p,q,c} :=< D_s, I_1^e, ..., I_p^e, I_1^h, \dots, I_q^h, I_1^{ch,Re}, \dots, I_c^{ch,Re}, I_1^{ch,Im}, \dots, I_c^{ch,Im} > .$$

§1.2.2: Model eigefunctions

An orthonormal basis of $L^2(S_L^1 \times \mathbb{R}^n)$ of joint eigenfunctions of $\mathcal{A}_{p,q,c}$ is given as follows: corresponding to the (p,q,c)- type of P_γ we can write

$$L^2(S_L^1 \times \mathbb{R}^n) = L^2(S_L^1) \otimes L^2(\mathbb{R}^p) \otimes L^2(\mathbb{R}^q) \otimes L^2(\mathbb{R}^{2c})$$

and construct the eigenfunctions as (tensor) products of the eigenfunctions on the factors. In the elliptic factors, the eigenfunctions are the normalized Hermite functions γ_q (cf.[F], or [Z.1] for a context similar to the one here). In the real hyperbolic factors,the action operators are the generators of the unitary dilations on $L^2(\mathbb{R})$ given by

$$U(\theta) f(x) = \theta^{\frac{1}{2}} f(\theta x), \qquad (\theta \in \mathbb{R}^+).$$

Their generalized eigenfunctions are the temperate distributions

$$x_+^{-\frac{1}{2}+ia}, \qquad x_-^{-\frac{1}{2}+ia}, \qquad (a \in \mathbb{R})$$

and any $f \in L^2(\mathbb{R}, dx)$ has the eigenfunction expansion

$$f(x) = \int_{\mathbb{R}} \hat{f}_+(a) x_+^{-\frac{1}{2}+ia} da + \int_{\mathbb{R}} \hat{f}_-(a) x_-^{-\frac{1}{2}+ia} da$$

with $\hat{f}_{\pm a} := \langle f, x_\pm^{-\frac{1}{2}+ia} \rangle$. In the complex hyperbolic (i.e. loxodromic) factors, the actions operators are given by the unitary dilations in polar coordinates

$$U(\rho) f(r, \theta) = \rho f(\rho r, \theta), \qquad (\rho \in \mathbb{R}^+)$$

together with rotations. The joint eigenfunctions are the temperate distributions on $\mathbb{R}^2$ given by

$$r^{it-1} e^{in\theta}, \qquad (t \in \mathbb{R})$$

and in a notation similar to that of the real hyperbolic case a function $f \in L^2(\mathbb{R}^2, rdrd\theta)$ may be expressed in the form

$$f(r,\theta) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}(t,\theta) r^{it-1} e^{in\theta} dt.$$

For future reference we summarize the situation in the following table.

Factor	Action	Eigenfunction
$L^2(S^1_L)$	D_s	$e^{is\frac{2\pi k}{L}}$
$L^2(\mathbb{R}^p)$	I^e_j	Hermite functions $\gamma_m, m \in \mathbb{N}^n$;
.	.	$\gamma_o(y) = \gamma_{iI}(y) := e^{-\frac{1}{2}\lvert y\rvert^2}$
.	.	$\gamma_m := C_m a_1^{\dagger m_1} ... a_n^{\dagger m_n} \gamma_o$
$L^2(\mathbb{R}^q)$	I^h_j	$\Pi^r_{j=1} y_{j\pm}^{ia_j - \frac{1}{2}}, a \in \mathbb{R}^r$
$L^2(\mathbb{R}^{2c})$	$I^{chRe}_j, I^{ch,Im}_j$	$\Pi^c_{j=1} r_j^{it_j-1} e^{in_j\theta_j}, t \in \mathbb{R}^c$

As in the introduction, we put:

$$H_{\alpha,\lambda,(\mu,\nu)} := [\sum_{j=1}^p \alpha_j \hat{I}^e_j + \sum_{j=1}^q \lambda_j \hat{I}^h_j + \sum_{j=1}^c \mu_j \hat{I}^{ch,Re}_j + \nu_j \hat{I}^{ch,Im}_j] \tag{1.2.2.1}$$

$$\mathcal{R} := \frac{1}{L}(LD_s + H_{\alpha,\lambda,(\mu,\nu)})$$

and note that
(1.2.2.2a)

$$\mathcal{R} e^{is\frac{2\pi k}{L}} \gamma_m(x) [\Pi^r_{j=1} y_{j\pm}^{ia_j-\frac{1}{2}}][\Pi^c_{j=1} r_j^{it_j-1} e^{in_j\theta_j}] = r_{kmnat} e^{is\frac{2\pi k}{L}} \gamma_m(x) [\Pi^r_{j=1} y_{j\pm}^{ia_j-\frac{1}{2}}][\Pi^c_{j=1} r_j^{it_j-1} e^{in_j\theta_j}]$$

with

$$r_{kmnat} = \frac{1}{L}(2\pi k + [\sum_{j=1}^p \alpha_j(m_j + \frac{1}{2}) + \sum_{j=1}^r \lambda_j a_j + \sum_{j=1}^c \mu_j t_j + \nu_j n_j]). \tag{1.2.2.2b}$$

2. The semi-classically scaled Laplacian

The significance of the maximal abelian algebra $\mathcal{A}_{pqc}$ will appear as soon as we semi-classically 'rescale' the Laplacian and conjugate the principal part, the 'linearized $\sqrt{\Delta}$', to its normal form.

Let us briefly recollect this rescaling, which proceeds exactly as in the purely elliptic case [Z.1,§2]. We first prepare the Laplacian by putting it in Fermi normal coordinates (s,y). It is then self-adjoint relative to the volume density $J(s,u)|ds||dy|$ in these coordinates. To simplify, we then conjugate it to the unitarily equivalent (1/2-density-) Laplacian

$$\Delta_{1/2} := J^{1/2} \Delta J^{-1/2},$$

which is self-adjoint with respect to the Lesbesgue density $|dsdy|$.

We thus have:

$$-\Delta_{1/2} = J^{-1/2}\partial_s g^{oo} J \partial_s J^{-1/2} + \sum_{ij=1}^n J^{-1/2} \partial_{y_i} g^{ij} J \partial_{y_j} J^{-1/2} \tag{2.1.1}$$

$$\equiv g^{oo}\partial_s^2 + \Gamma^o \partial_s + \sum_{ij=1}^n g^{ij} \partial_{u_i} \partial_{y_j} + \sum_{i=1}^n \Gamma^i \partial_{y_i} + \sigma_o.$$

Semi-classical rescaling then involves two conjugations: First, by $M_h =$ multiplication by $e^{\frac{is}{Lh}}$,

$$-M_h^* \Delta M_h = -(hL)^{-2} g^{oo} + 2i(hL)^{-1} g^{oo} \partial_s + i(hL)^{-1} \Gamma^o + \Delta$$

and then by the semi-classical dilation $T_h f(s,y) = f(s, h^{-\frac{1}{2}}y)$. The complete conjugation $-T_h^* M_h^* \Delta M_h T_h$ results in the *semi-classically scaled Laplacian*

$$(2.1,2)\ \ -\Delta_h = -(hL)^{-2} g^{oo}_{[h]} + 2i(hL)^{-1} g^{oo}_{[h]}\partial_s + i(hL)^{-1}\Gamma^{o}_{[h]} + h^{-1}(\sum_{ij=1}^{n} g^{ij}_{[h]}\partial_{y_i}\partial_{y_j}) + h^{-\frac{1}{2}}(\sum_{i=1}^{n}\Gamma^{i}_{[h]}\partial_{y_i}) + (\sigma)_{[h]},$$

the subscript $[h]$ indicating to dilate the coefficients of the operator in the form, $f_h(s,y) := f(s, h^{\frac{1}{2}}y)$.

Expanding the coefficients in Taylor series at $h = 0$, we obtain the semi-classical expansion

$$(2.1.3)\qquad \Delta_h \sim \sum_{m=0}^{\infty} h^{(-2+m/2)}\mathcal{L}_{2-m/2}$$

where $\mathcal{L}_2 = L^{-2}$, $\mathcal{L}_{3/2} = 0$ and where

$$(2.1.4).\qquad \mathcal{L}_1 = 2L^{-1}[i\frac{\partial}{\partial s} + \frac{1}{2}\{\sum_{j=1}^{n}\partial^2_{y_j} - \sum_{ij=1}^{n} K_{ij}(s) y_i y_j\}]$$

We will denote the bracketed operator, the 'linearized $\sqrt{\Delta}$' by $\mathcal{L}$. It is of order 1 in the sense of pseudo-differential operators (using the Weyl filtration in the transverse variables) and as will be seen below is the principal term in the semi-classical expansion of the square root of the rescaled Laplacian.

(2.1.A) Appendix on metric scaling

In addition to semi-classical scaling, we have also just introduced an independent scaling, *metric scaling*, which has to do with the behaviour of objects under dilations $g \to \epsilon^2 g$ of the metric. As discussed in detail in [Z.1], the wave invariants have well-defined weights under metric rescaling and in analysing them it is very convenient to rescale all objects to be weightless. For instance, as discussed in [Z.1, §1.4], an ω- symplectic basis of Jacobi fields has weight $\frac{1}{2}$ and its time derivative has weight $-\frac{1}{2}$. To render it weightless a Jacobi eigenfield Y should be replaced by $L^{-\frac{1}{2}}Y$, $\dot{Y}$ by $L^{\frac{1}{2}}\dot{Y}$ etc. The resulting weightless Wronskian matrix is then denoted by $\mathcal{W}_L$. It is essentially the weightless matrix denoted $\mathcal{A}_L$ in [Z.1].

To render the coordinates (y,η) weightless under metric rescaling, we also change variables to $x = L^{-1}y$ and rewrite Δ_h and the $\mathcal{L}_{2-\frac{n}{2}}$'s in terms of the x-variables. For instance, $\mathcal{L}$ then takes the form:

$$\mathcal{L} = i\frac{\partial}{\partial s} + \frac{1}{2}[\sum_{j=1}^{n} L^{-1}\partial^2_{x_j} - \sum_{ij=1}^{n} LK_{ij}(s)x_i x_j].$$

The symplectic coordinates on the symplectic normal space $T^*\mathbb{R}^n$ to $\mathbb{R}^+\gamma$ will henceforth be denoted (x,ξ).

For the sake of brevity we will not draw much attention to metric scaling in the various steps to come in the normal form algorithm. In all cases, the role of metric scaling is identical to that in the elliptic case [Z.1].

§2.2: Conjugating Δ_h to the model

We now conjugate the semi-classically scaled Laplacian Δ_h from $L^2(N_\gamma)$ to the model $L^2(S^1_L \times \mathbb{R}^n)$ by means of the moving metaplectic operator $\mu(\mathcal{W}_L)$,

$$\mu(\mathcal{W}_L)f(s,y) := \mu(\mathcal{W}_L(s))f(s,y).$$

The motivation for this conjugation comes from:

(2.2.1) Proposition

$$\mathcal{L} = \mu(\mathcal{W}_L^*)D_s\mu(\mathcal{W}_L^*)^{-1}$$

where $\mathcal{W}_L$ is the weightless Wronskian matrix and μ is the metaplectic representation.

Proof:

First, let us ignore the scaling parameter L, i.e. let us put $L = 1$. The right side is then equal to $(D_s + \mu(\mathcal{W}(s))^* D_s \mu(\mathcal{W}(s))$. To evaluate the second term, we recall that the columns of $\mathcal{W}$ are Jacobi fields, and that Jacobi's equation is equivalent to the linear system $\frac{D}{ds}(Y,P) = JH(Y,P)$ with $P = \frac{DY}{ds}$, and with

$$H = \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix}.$$

Hence, the second term is $\frac{1}{i} d\mu(JH)$ with $d\mu$ the derived metaplectic representation. But $\frac{1}{i} d\mu(JH) = 1/2(\sum_{i=1}^n \partial_{y_i}^2 - \sum_{ij=1}^n K_{ij}(s) y_i y_j)$ [F]. Re-inserting L to make all objects weightless, we get the formed claimed above. □

Thus, conjugation by $\mathcal{W}_L$ puts the principal term $\mathcal{L}$ of Δ_h into the simple normal form D_s. This suggests conjugating the full rescaled Laplacian by $\mu(\mathcal{W}_L)$ to the 'twisted model' semi-classical Laplacian

(2.2.2)
$$\mathcal{D}_h = \mu(\mathcal{W}_L^*)^{-1} \Delta_h \mu(\mathcal{W}_L^*)$$

which has the asymptotic expansion

(2.2.3)
$$\mathcal{D}_h \sim \sum_{m=o}^{\infty} h^{(-2+\frac{m}{2})} \mathcal{D}_{2-\frac{m}{2}}$$

with $\mathcal{D}_2 = I, \mathcal{D}_{\frac{3}{2}} = 0, \mathcal{D}_1 = D_s$. Thus, $\mathcal{D}_h$ is a small perturbation of D_s, and one may expect that perturbation theory can be used to find a good normal form for the whole of $\mathcal{D}_h$.

Before doing so, we must consider which Hilbert space is the natural domain for $\mathcal{D}_h$. The point is that the conjugation has non-trivial monodromy (§1.1b) and hence the conjugate will act on functions transforming correctly under the monodromy group.

We can describe the Hilbert space in terms of quantum mapping cylinders [Z.1]. First, we consider the holonomy aspect, put

(2.2.4)
$$C_T^\infty(\mathbb{R} \times \mathbb{R}^n) := \{f \in C^\infty(\mathbb{R} \times \mathbb{R}^n) : f(s+L,u) = \mu(T) f(s,u)\}$$

and let $\mathcal{H}_T$ denote its closure with respect to the obvious inner product over $[0,L) \times \mathbb{R}^n$. Note that the metaplectic operator $\mu(T)$ is simply

$$\mu(T) f(u) = f(t^{-1} u)$$

and hence that

$$C_T^\infty(\mathbb{R} \times \mathbb{R}^n) \sim C^\infty(N_\gamma)$$

where the isomorphism is simply the pull-back by the exponential map defined by the frame $e(s)$. In other words, expressed in terms of Fermi coordinates relative to a normal frame, $L^2(N_\gamma)$ becomes the quantum mapping cylinder of $\mu(T^*)$. Let us note however that Δ_h and hence all the $\mathcal{L}_{2-\frac{k}{2}}$'s are invariant under under $\mu(T)$, so that it will play an insignificant role for our purposes.

On the other hand, the quantized linear Poincare map $\mu(P_\gamma)$ will play an essential role. Hence we introduce its quantized mapping cylinder

(2.2.5)
$$\mathcal{H}_\gamma := \{f \in L^2_{loc}(\mathbb{R} \times \mathbb{R}^n) : \tau_L f = \mu(P_\gamma) f\}$$

and note that

$$\mu(\mathcal{W}_L) : L^2(N_\gamma) \to \mathcal{H}_\gamma$$

is a unitary equivalence. Hence, the natural domain for $\mathcal{D}_h$ is the quantum mapping cyliner of $\mu(P_\gamma)$.

In the calculation of traces, it is simpler to work in the original model $L^2(S_L^1 \times \mathbb{R}^n)$. Hence we will also consider the conjugate of $\mathcal{D}_h$ under a conjugation which untwists the mapping cylinder of $\mu(P_\gamma)$. That is, we connect P_γ to the identity by a segment of the one-parameter subgroup $P_\gamma(s)$ thru I and P_γ, which exists by our non-degeneracy assumption on P_γ. Indeed, after diagonalizing P_γ and consulting the list of symplectic equivalence classes of quadratic forms, we see that

$$P_\gamma = exp(\Xi_{H_{\alpha,\lambda,(\mu,\nu)}})$$

where Ξ_f denotes the Hamilton vector field of f and where $exp \circ \Xi$ denotes the exponential map from $sp(n,\mathbb{R}) \to Sp(n,R)$, with $sp(n,\mathbb{R})$ viewed as the Poisson algebra of quadratic functions on $\mathbb{R}^{2n}$. We then have

$$P_\gamma(s) = exp(s\Xi_{H_{\alpha,\lambda,(\mu,\nu)}})$$

and quantize this subgroup as

$$\mu(P_\gamma(s)) = e^{isH_{\alpha,\lambda,(\mu,\nu)}} = e^{is[\sum_{j=1}^p \alpha_j I_j^e + \sum_{j=1}^q \lambda_j I_j^h + \sum_{j=1}^c \mu_j I_j^{ch,Re} + \nu_j I_j^{ch,Im}]}. \tag{2.2.6}$$

Conjugation by $\mu(P_\gamma)$ transforms $\mathcal{D}_h$ into the model semi-classically scaled Laplacian

$$\mathcal{R}_h := \mu(P_\gamma)\mu(\mathcal{W}_L)\Delta_h \mu(\mathcal{W}_L)^* \mu(P_\gamma)^* \sim \sum_{m=o}^{\infty} h^{(-2+\frac{m}{2})}\mathcal{R}_{2-\frac{m}{2}} \tag{2.2.7}$$

with $\mathcal{R}_2 = I, \mathcal{R}_{\frac{3}{2}} = 0$, and with $\mathcal{R}_1 := \mathcal{R}$. All coefficients of terms in $\mathcal{R}$ are periodic in s and have weight -2 under metric rescaling.

3. Semi-classical normal form

We now wish to put $\mathcal{R}_h$ into semi-classical normal form, in the sense of [Z.1, Lemma 2.22]. This is the key transitional step in putting Δ into microlocal normal form and is the source of the connections to local geometric invariants. The method is essentially the same as in the elliptic case, both in method and in detail. We therefore present only the first two steps in the proof and refer to [Z.1, loc.cit] for the inductive argument.

As in the elliptic case, we state the result in terms of the $\mathcal{R}$-operators since the trace will later be analysed in this model. However, most of the proof will take place in the twisted model, where the 'linearized Laplacian' is D_s and the equations simplify most. In the following, the notation $|_o$ means to restrict to functions in the kernel of $\mathcal{R}$, that is, to elements of $\mathcal{R}$-weight zero. In the twisted model, these are simply functions independent of s. In the passage from the semi-classical normal form to the microlocal (quantum Birkhoff) normal form, the various operators will only be applied to such weightless elements. This explains the rather complicated statement to follow; the result is only simple and natural when restricted to elements of weight zero.

(3.1) **Lemma (cf. [Z.1, Lemma 2.22]** *There exists an L-dependent h-pseudodifferential operator $W_h = W_h(s,x,D_x)$ on $L^2(S^1_L \times \mathbb{R}^n)$ such that, for each $s \in S^1_L$,*

$$W_h(s,x,D_x) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is unitary, and such that

$$W_h^* \mathcal{R}_h W_h \sim -h^{-2}L^{-2} + 2h^{-1}L^{-1}\mathcal{R} + \sum_{j=0}^{\infty} h^{\frac{j}{2}}\mathcal{R}^{\infty}_{2-\frac{j}{2}}(s,D_s,x,D_x)$$

where

(i) $\mathcal{R}^{\infty}_{2-\frac{j}{2}}(s,D_s,x,D_x) = \mathcal{R}^{\infty,2}_{2-\frac{j}{2}}\mathcal{R}^2 + \mathcal{R}^{\infty,1}_{2-\frac{j}{2}}\mathcal{R} + \mathcal{R}^{\infty,o}_{2-\frac{j}{2}}$, with $\mathcal{R}^{\infty,k}_{2-\frac{j}{2}} \in C^\infty(S^1_L, \mathcal{E}^{j-2k}_\epsilon)$;

(ii) $\mathcal{R}^{\infty}_{2-j}(s,D_s,x,D_x)|_o = \mathcal{R}^{\infty,o}_{2-j}(s,x,D_x)|_o = f_j(I^e_1,\dots,I^e_p,I^h_1,\dots,I^h_r,I^{chRe}_1,\dots I^{chRe}_c,I^{chIm}_1,\dots,I^{chIm}_c)|_o$ for certain polynomials f_j of degree j+2 on $\mathbb{R}^n$, i.e. $f_j(I^e_1,\dots,I^e_p,I^h_1,\dots,I^h_r,I^{chRe}_1,\dots I^{chRe}_c,I^{chIm}_1,\dots,I^{chIm}_c) \in \mathcal{P}^{j+2}_{\mathcal{I}}$

(iii) $\mathcal{R}^{\infty}_{2-\frac{2k+1}{2}}(s,D_s,x,D_x)|_o = \mathcal{R}^{\infty,o}_{2-\frac{2k+1}{2}}(s,x,D_x)|_o = 0$;

(iv) The terms I^e_j, I^h_j, I^{ch}_j are weightless under metric scalings and all of the $\mathcal{R}$'s have weight -2.

Proof:

As in the elliptic case [Z.1,Lemma 2.22], the operator W_h will be constructed as the asymptotic product

$$W_h := \mu(P_\gamma)^* \circ \Pi_{k=1}^{\infty} W_{h^{\frac{k}{2}}} \circ \mu(P_\gamma) \tag{3.2}$$

of weightless unitary h-pseudodifferential operators on $\mathbb{R}^n$, with

$$W_{h^{\frac{k}{2}}} := exp(ih^{\frac{k}{2}}Q_{\frac{k}{2}}) \tag{3.3}$$

and with $h^{\frac{k}{2}}Q_{\frac{k}{2}} \in h^{\frac{k}{2}}\mathbb{C}^\infty(S^1_L) \otimes \mathcal{E}^{k+2}$ of total order 1. The product will converge, for each s, to a unitary operator in $\Psi^o_h(\mathbb{R}^n)$ (we refer to [Sj] for a discussion of such asymptotic products).

We first construct a weightless $Q_{\frac{1}{2}}(s, x, D_x) \in C^\infty(S^1_L) \otimes \mathcal{E}^3_\epsilon$ such that

$$e^{-ih^{\frac{1}{2}}Q_{\frac{1}{2}}}\mathcal{R}_h e^{ih^{\frac{1}{2}}Q_{\frac{1}{2}}}|_o = [-h^{-2}L^{-2} + 2h^{-1}L^{-1}\mathcal{R} + \mathcal{R}_o^{\frac{1}{2}} + \dots]|_o \tag{3.4a}$$

where the dots $\dots$ indicate higher powers in h. The operator $Q_{\frac{1}{2}}$ then must satisfy the commutation relation

$$\{[L^{-1}\mathcal{R}, Q_{\frac{1}{2}}] + \mathcal{R}_{\frac{1}{2}}\}|_o = 0. \tag{3.4b}$$

To solve for $Q_{\frac{1}{2}}$, we conjugate back to the $\mathcal{D}_{2-\frac{m}{2}}$'s of the twisted model by $\mu(P_\gamma)$, which transforms $\mathcal{R}$ into D_s. The commutation relation thus becomes

$$\{[L^{-1}D_s, \mu(P_\gamma))^*Q_{\frac{1}{2}}\mu(P_\gamma)] + \mathcal{D}_{\frac{1}{2}}\}|_o = 0, \tag{3.4c}$$

that is,

$$L^{-1}\partial_s\{\mu(P_\gamma)^*Q_{\frac{1}{2}}\mu(P_\gamma))\}|_o = -i\{\mathcal{D}_{\frac{1}{2}}\}|_o \tag{3.4d}$$

where $\partial_s A$ is the Weyl operator whose complete symbol is the s-derivative of that of A. Since (3.4d) is simpler than (3.4b), we henceforth conjugate everything by $\mu(P_\gamma))$, and relabel the operators $\mu(P_\gamma)^*Q\mu(P_\gamma)$ by $\tilde{Q}$. The resulting $\mathcal{D}$'s then transform under τ_L like operators on the quantum mapping cylinder of $\mu(P_\gamma)$. Our problem is thus to solve (3.4d) with an operator $\tilde{Q}_{\frac{1}{2}}$ satisfying

$$\tau_L\tilde{Q}_{\frac{1}{2}}\tau_L^* = \mu(P_\gamma)\tilde{Q}_{\frac{1}{2}}\mu(P_\gamma)^*.$$

To solve the equation (3.4d) we rewrite it in terms of complete Weyl symbols. We will use the notation $A(s, x, \xi)$ for the complete Weyl symbol of the operator $A(s, x, D_x)$. Then (3.4d) becomes

$$L^{-1}\partial_s\tilde{Q}_{\frac{1}{2}}(s, x, \xi) = -i\mathcal{D}_{\frac{1}{2}}|_o(s, x, \xi) \tag{3.5a}$$

with

$$\tilde{Q}_{\frac{1}{2}}(s + L, x, \xi) = \tilde{Q}_{\frac{1}{2}}(s, P_\gamma(x, \xi)).$$

We solve (3.5a) with the Weyl symbol

$$\tilde{Q}_{\frac{1}{2}}(s, x, \xi) = \tilde{Q}_{\frac{1}{2}}(0, x, \xi) + L\int_0^s -i\mathcal{D}_{\frac{1}{2}}|_o(u, x, \xi)du$$

where $\tilde{Q}_{\frac{1}{2}}(0, x, \xi)$ is determined by the consistency condition

$$\tilde{Q}_{\frac{1}{2}}(L, x, \xi) - \tilde{Q}_{\frac{1}{2}}(0, x, \xi) = L\int_0^L -i\mathcal{D}_{\frac{1}{2}}|_o(u, x, \xi)du \tag{3.5b}$$

or in view of the periodicity condition in (3.5a),

$$\tilde{Q}_{\frac{1}{2}}(0, P_\gamma(x, \xi)) - \tilde{Q}_{\frac{1}{2}}(0, x, \xi) = L\int_0^L -i\mathcal{D}_{\frac{1}{2}}|_o(u, x, \xi)du. \tag{3.5c}$$

To solve, we use that $\mathcal{D}_{\frac{1}{2}}|_o(u, x, \xi)$ is a polynomial of degree 3 in (x, ξ). It will be most convenient to express this polynomial in coordinates relative to the eigenvectors of the Poincare map. In the elliptic planes, we use the complex coordinates $z_j = x_j + i\xi_j$ and $\bar{z}_j = x_j - i\xi_j$ $(j = 1, \dots, p)$ in which the action of P_γ is diagonal. In the real hyperbolic planes we use the real coordinates $(y_j, \eta_j) = (x_j, \xi_j), (j = p+1, \dots p+q)$ in which the real hyperbolic part of P_γ is diagonal. Finally, in the complex hyperbolic (loxodromic) 4-spaces we use the coordinates $w_j = x_j + ix_{c+j}, \bar{w}_j = x_j - ix_{c+j}, \omega_j = \xi_j - i\xi_{c+j}, \bar{\omega}_j = \xi_j + i\xi_{c+j}, (j = p+q+1, \dots p+q+c)$ in which the complex hyperbolic part of P_γ is diagonal.

We will denote the Weyl symbols in these coordinates by their previous expressions. We also suppress the subscripts by using vector notation $z, \bar{z}, y, \eta, w, \bar{w}, \omega, \bar{\omega}$. Thus, (3.5c) becomes

$$\tilde{Q}_{\frac{1}{2}}(0, e^{i\alpha}z, e^{-i\alpha}\bar{z}, e^\lambda y, e^{-\lambda}\eta, e^{\mu+i\nu}w, e^{\mu-i\nu}\bar{w}, e^{-\mu+i\nu}\omega, e^{-\mu-i\nu}\bar{\omega}) - \tilde{Q}_{\frac{1}{2}}(0, z, \bar{z}, y, \eta, w, \bar{w}, \omega, \bar{\omega}) =$$

$$(3.6) \qquad = L\int_0^L -i\mathcal{D}_{\frac{1}{2}}|_o(u,z,\bar z,y,\eta,w,\bar w,\omega,\bar\omega)du.$$

We now use that $\mathcal{D}_{\frac{1}{2}}(u,z,\bar z,y,\eta,w,\bar w,\omega,\bar\omega)$ is a polynomial of degree 3 to solve (3.5c). If we put

$$(3.7a) \qquad \tilde Q_{\frac{1}{2}}(s,z,\bar z,y,\eta,w,\bar w,\omega,\bar\omega) := \sum_{|a|+|\bar a|+|b|+|c|+|\bar c|\le 3} q_{\frac{1}{2};a\bar a bc\bar c}(s) z^a \bar z^{\bar a} y^{b_1}\eta^{b_2} w^{c_1}\omega^{c_2}\bar w^{\bar c_1}\bar\omega^{\bar c_2}$$

and

$$(3.7b) \qquad \mathcal{D}_{\frac{1}{2}}|_o(s,z,\bar z,y,\eta,w,\bar w,\omega,\bar\omega)du := \sum_{|a|+|\bar a|+|b|+|c|+|\bar c|\le 3} d_{\frac{1}{2};a\bar a bc\bar c}(s) z^a \bar z^{\bar a} y^{b_1}\eta^{b_2} w^{c_1}\omega^{c_2}\bar w^{\bar c_1}\bar\omega^{\bar c_2}$$

then (3.6) becomes

$$\sum_{|a|+|\bar a|+|b|+|c|+|\bar c|\le 3} (1-e^{i(a-\bar a)\alpha + i(c_1-\bar c_1)\nu + (b_1-b_2)\lambda + (c_2-\bar c_2)\mu}) q_{\frac{1}{2};a\bar a bc\bar c}(0) z^a \bar z^{\bar a} y^{b_1}\eta^{b_2} w^{c_1}\omega^{c_2}\bar w^{\bar c_1}\bar\omega^{\bar c_2} =$$

$$(3.8). \qquad = -iL^2 \sum_{|a|+|\bar a|+|b|+|c|+|\bar c|\le 3} \bar d_{\frac{1}{2};a\bar a bc\bar c} z^a \bar z^{\bar a} y^{b_1}\eta^{b_2} w^{c_1}\omega^{c_2}\bar w^{\bar c_1}\bar\omega^{\bar c_2}$$

Under the non-degeneracy assumption on P_γ, we can solve with

$$(3.9) \qquad q_{\frac{1}{2};a\bar a bc\bar c}(0) = -iL^2(1-e^{i(a-\bar a)\alpha + i(c_1-\bar c_1)\nu + (b_1-b_2)\lambda + (c_2-\bar c_2)\mu})^{-1} d_{\frac{1}{2};a\bar a bc\bar c}$$

since $i(a-\bar a)\alpha + i(c_1-c_2)\nu) + (b_1-b_2)\lambda + (\bar c_1 - \bar c_2)\mu) = 2\pi i k$ only if $a=\bar a, b_1=b_2, c=\bar c$ and there are no such $(a,\bar a, b_1, b_2, c, \bar c)$ in an odd-index equation.

Precisely as in the purely elliptic case of [Z.1], we see that $\tilde Q_{\frac{1}{2}}$ is a pseudodifferential operator on $\mathbb{R}^n$ with the same order, same order of vanishing, and same parity as the restriction of $\mathcal{D}_{\frac{1}{2}}$ to elements of weight zero. We then extend it as a pseudodifferential operator of the form

$$\tilde Q_{\frac{1}{2}} \in \Psi^o(\mathbb{R}^1)\otimes \mathcal{E}^3_\epsilon$$

on all of $\mathcal{H}_\gamma$ by decreeing that it commute with s. The conjugate by $\mu(P_\gamma)$ then defines a unitary operator $W_{h\frac{1}{2}} \in \Psi^o_h(S^1\times\mathbb{R}^n)$ satisfying (3.4a). The corresponding twisted unitary operator with exponent $\tilde Q_{\frac{1}{2}}$, i.e. the image of $W_{h\frac{1}{2}}$ under conjugation by $\mu(P_\gamma)$, will be denoted $\tilde W_{h\frac{1}{2}}$.

The effect of this first conjugation is precisely as in the elliptic case: Since $h^{\frac{1}{2}}\tilde Q_{\frac{1}{2}}$ is of total order 1, $h^{\frac{1}{2}} ad(\tilde Q_{\frac{1}{2}})$ (with $ad(A)B := [B,A]$) preserves the total order in $\Psi_h^{(*,*,*)}$, and hence $\tilde W_{h\frac{1}{2}}$ is an order-preserving automorphism of the model pseudodifferential algebra. It is moreover independent of D_s and has an odd polynomial Weyl symbol, so that

$$(3.10) \qquad h^{\frac{1}{2}} ad(\tilde Q_{\frac{1}{2}}) : h^{\frac{k}{2}}\Psi^l(\mathbb{R})\otimes\mathcal{E}^m_\epsilon \to h^{\frac{k+1}{2}}[\Psi^{l-1}(\mathbb{R})\otimes\mathcal{E}^{m+3}_\epsilon + \Psi^l(\mathbb{R})\otimes\mathcal{E}^{m+1}_\epsilon].$$

Finally, the $d_{\frac{1}{2};m,n}$'s have weight -2, the variables z have weight 0 and hence the $q_{\frac{1}{2};m,n}$'s have weight 0.

Consider now the element

$$\mathcal{D}_h^{\frac{1}{2}} := \tilde W^*_{h\frac{1}{2}}\mathcal{D}_h \tilde W_{h\frac{1}{2}} \in \Psi^2_h(\mathbb{R}^1\times\mathbb{R}^n)$$

which can be expanded in the semi-classical series

$$(3.11) \qquad \mathcal{D}_h^{\frac{1}{2}} \sim \sum_{n=o}^\infty h^{-2+\frac{n}{2}} \sum_{j+m=n} \frac{i^j}{j!}(ad\tilde Q_{\frac{1}{2}})^j \mathcal{D}_{2-\frac{m}{2}}$$

$$:= h^{-2}L^{-2} + h^{-1}L^{-1}D_s + \sum_{n=3}^\infty h^{-2+\frac{n}{2}}\mathcal{D}^{\frac{1}{2}}_{2-\frac{n}{2}}.$$

An obvious induction as in the elliptic case [loc.cit.] gives that

$$ad(\tilde Q_{\frac{1}{2}})^j\mathcal{D}_{2-\frac{m}{2}} \in C^\infty(\mathbb{R},\mathcal{E}^{m+j-4}_\epsilon)D_s^2 + C^\infty(\mathbb{R},\mathcal{E}^{m+j-2}_\epsilon)D_s + C^\infty(\mathbb{R},\mathcal{E}^{m+j}_\epsilon).$$

It follows that $\mathcal{D}^{\frac{1}{2}}_{2-\frac{n}{2}}$ has the same filtered structure as $\mathcal{D}_{2-\frac{n}{2}}$.

We carry this procedure out one more step before referring to [Z.1] for the inductive argument, since the even steps behave differently from the odd ones. We thus seek an element $\tilde{Q}_1(s,x,D_x) \in \Psi^*(S^1 \times \mathbb{R}^n)$ and an element $f_o(I_1^e,\dots,I_p^e,I_1^h,\dots,I_r^h,I_1^{ch,Re},I_1^{ch,Im},\dots,I_c^{ch,Re},I^{ch,Im}) \in \mathcal{A}$ so that

$$\mathcal{D}_h^1 := \tilde{W}_{h1}^* \mathcal{D}^{\frac12} \tilde{W}_{h1} = h^{-2}L^{-2} + h^{-1}L^{-1}D_s + h^{-\frac12}\mathcal{D}_{\frac12}^{\frac12} + \mathcal{D}_o^1(s,D_s,x,D_x) + \dots \tag{3.12}$$

with

$$\mathcal{D}_o^1(s,D_s,x,D_x)|_o = f_o(I_1^e,\dots,I_c^{ch,Im}) \tag{3.13a}$$

with $\tilde{W}_{h1} = e^{ih\tilde{Q}_1}$, and where the dots signify terms of higher order in h. Note that $\mathcal{D}_{\frac12}^1 = \mathcal{D}_{\frac12}^{\frac12}$, so that (3.12) implies that

$$\{h^{\frac12}\mathcal{D}_{\frac12}^1 + \mathcal{D}_o^1\}|_o = f_o(I_1^e,\dots,I_c^{ch,Im}). \tag{3.13b}$$

The condition on $\tilde{Q}_1$ is then

$$\{[D_s,\tilde{Q}_1] + \mathcal{D}_o^{\frac12}\}|_o = f_o(I_1^e,\dots,I_c^{ch,Im}) \tag{3.14a}$$

or equivalently

$$\partial_s\tilde{Q}_1|_o = \{-\mathcal{D}_o^{\frac12} + f_o(I_1^e,\dots,I_c^{ch,Im})\}|_o \tag{3.14b).}$$

We solve (3.14b) by again expressing everything in terms of complete Weyl symbols relative to the eigenvector coordinates. Thus we rewrite (3.14b)) in the form

$$L^{-1}\partial_s\tilde{Q}_1(s,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega}) =$$

$$= -i\{\mathcal{D}_o^{\frac12}|_o(s,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega}) - f_o(|z_1|^2,\dots,|z_p|^2,y_1\eta_1,\dots,y_r\eta_r,Rew\omega,Imw\omega)\} \tag{3.15a}$$

or equivalently

$$\tilde{Q}_1(s,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega}) = \tilde{Q}_1(0,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega})$$

$$-iL\int_0^s[\mathcal{D}_o^{\frac12}|_o(u,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega}) - f_o(|z_1|^2,\dots,|z_p|^2,y_1\eta_1,\dots,y_r\eta_r,Rew\omega,Imw\omega)]du \tag{3.15b}$$

and solve simeltaneously for $\tilde{Q}_1$ and f_o. The consistency condition determining a unique solution is that

$$\tilde{Q}_1(L,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega}) = \tilde{Q}_1(0,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega})$$

$$-iL\int_0^L[\mathcal{D}_o^{\frac12}|_o(u,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega}) - f_o(|z_1|^2,\dots,|z_p|^2,y_1\eta_1,\dots,y_q\eta_q,Rew\omega,Imw\omega)]du. \tag{3.16a}$$

or in view of the twisted periodicity condition

$$\tilde{Q}_1(0,e^{i\alpha}z,e^{-i\alpha}\bar{z},e^{\lambda}y,e^{-\lambda}\eta,e^{\mu+i\nu}w,e^{\mu-i\nu}\omega,e^{-\mu+i\nu}\bar{w},e^{-\mu-i\nu}\bar{\omega}) - \tilde{Q}_1(0,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega}) =$$

$$-iL\{\int_0^L \mathcal{D}_o^{\frac12}|_o(u,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega})du - Lf_o(|z_j|^2,y_j\eta_j,Rew_m\omega_m,Imw_n\omega_n)\}. \tag{3.16b}$$

In the spirit of the previous step, we use that $\mathcal{D}_o^{\frac12}|_o(u,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega})$ is a polynomial of degree 4 to solve the equation. We put

$$\tilde{Q}_1(s,z,\bar{z},y,\eta,w,\omega,\bar{w},\bar{\omega}) = \sum_{|a|+|\bar{a}|+|b|+|c|+|\bar{c}|\le 4} q_{1;a\bar{a}bc\bar{c}}(s)z^a\bar{z}^{\bar{a}}y^{b_1}\eta^{b_2}w^{c_1}\omega^{c_2}\bar{w}^{\bar{c}_1}\bar{\omega}^{\bar{c}_2},$$

and in an abbreviated notation,

$$f_o(|z|^2,y\cdot\eta,Rew\cdot\omega,Imw\cdot\omega) = \sum_{|k|+|\ell|+|n|\le 2} c_{ok\ell n}|z|^{2k}(y\cdot\eta)^{\ell}(Rew\cdot\omega)^{n_1}(Imw\cdot\omega)^{n_2}, \tag{3.17}$$

and

$$\mathcal{D}_o^{\frac12}|_o(s,z,\bar{z},y,\eta)du := \sum_{|a|+|\bar{a}|+|b|+|c|+|\bar{c}|\le 4} d^{\frac12}_{o;a\bar{a}bc\bar{c}}(s)z^a\bar{z}^{\bar{a}}y^{b_1}\eta^{b_2}w^{c_1}\omega^{c_2}\bar{w}^{\bar{c}_1}\bar{\omega}^{\bar{c}_2},$$

and finally

$$\bar{d}^{\frac{1}{2}}_{o;a\bar{a}bc\bar{c}} := \frac{1}{L}\int_o^L d^{\frac{1}{2}}_{o;a\bar{a}bc\bar{c}}(s)ds$$

As above, we can solve for the off-diagonal coefficients where either $a \neq b$ or $m \neq n$

$$q_{1;a\bar{a}bc\bar{c}}(0) = -iL^2(1 - e^{i(a-\bar{a})\alpha + i(c_1-\bar{c}_1)\nu + (b_1-b_2)\lambda) + (c_2-\bar{c}_2)\mu})^{-1}\bar{d}^{\frac{1}{2}}_{o;a\bar{a}bc\bar{c}} \tag{3.18a}$$

and must set the diagonal coefficients with $a = \bar{a}, c_1 = \bar{c}_1, b_1 = b_2, c_2 = \bar{c}_2$ equal to zero. The expression in (3.18a) is well-defined by the non-degeneracy assumption. The coefficients $c_{ok\ell}$ are then determined by

$$c_{ok\ell n} = \bar{d}^{\frac{1}{2}}_{1;kk\ell\ell nn}. \tag{3.18b}$$

It is evident that $\tilde{Q}_1$ and $f_o(I_1^e, \dots, I_c^{ch,Im})$ are even polynomial pseudodifferential operators of degree 4 in the variables (x, D_x), that $\tilde{Q}_1$ is weightless under metric rescalings and that the coefficients $c_{ok\ell n}$ are of weight -2.

The rest proceeds as in the elliptic case. □

4. Normal form of the Laplacian: Proof of Theorem I

We now use the semi-classical normal forms to put the Laplacian into quantum Birkhoff normal form. Essentially this amounts to taking direct sums (or integrals) of the semi-classical normal form over various internal Planck constants.

Proof of Theorem I: As in the elliptic case, we make the transition from the semi-classical normal form to the quantum Birkhoff normal form by using genreralized eigenfunction expansions for the model algebra.

From the table in §1.2.1 we see that a function $f \in L^2(S^1_L \times \mathbb{R}^p_x \times \mathbb{R}^q_y \times \mathbb{R}^{2c}_{r,\theta})$ can be expanding in terms of joint $\mathcal{A}_{pqc}$-eigenfunctions as:

$$f(s,x,y,r,\theta) = \sum_{\pm}\sum_{(k,m,n)\in\mathbf{N}^{1+p+c}}\int_{\mathbb{R}^q}\int_{\mathbb{R}^{+c}} \hat{f}_{\pm}(k,m,n,a,t)e^{ir_{kmnat}}e^{i<n,\theta>}\gamma_m(x)y_{\pm}^{ia-\frac{1}{2}}r^{it-1}dadt.$$

Here as in §3, we have used the notation x for linear coordinates on the elliptic factors, y for those on the real hyperbolic factors, and polar coordinates $w_j = r_je^{i\theta_j}$ in each w_j-plane of the complex hyperbolic factors. We also employ a multi-index notation.

We now assemble the semi-classical intertwining operators into the Fourier-Hermite-Mellin -series-integral intertwining operator

$$W_\gamma : L^2(S^1_L \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{2c}, dsdxdydw), \to L^2(S^1_L \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{2c}, dsdxdydw) \tag{4.1}$$

$$W_\gamma\sum_{\pm}\sum_{(k,m,n)\in\mathbf{N}^{1+p+c}}\int_{\mathbb{R}^q}\int_{\mathbb{R}^{+c}}\hat{f}(k,m,n,a,t)e^{ir_{kmnat}s}e^{i<n,\theta>}\gamma_m(x)y_{\pm}^{ia-\frac{1}{2}}r^{it-1}dadt =$$

$$\sum_{\pm}\sum_{(k,m,n)\in\mathbf{N}^{1+p+c}}\int_{\mathbb{R}^q}\int_{\mathbb{R}^{+c}}\hat{f}(k,m,n,a,t)e^{ir_{kmnrt}s}W_{kmnrt}e^{i<n,\theta>}\gamma_m(x)y_{\pm}^{ir-\frac{1}{2}}r^{it-1}dadt$$

with

$$W_{kmnrt} := \mu(\tilde{\mathcal{W}}(s)^*)W_{r^{-1}_{kqnat}}\mu(\tilde{\mathcal{W}}_s)^{*-1}.$$

Also, the dilation operators will be assembled into the operator

$$T : L^2(S^1_L \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{2c}, dsdxdydw) \to L^2(S^1_L \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{2c}, dsdxdydw)), \tag{4.2}$$

$$T\sum_{\pm}\sum_{(k,m,n)\in\mathbf{N}^{1+p+c}}\int_{\mathbb{R}^q}\int_{\mathbb{R}^{+c}}\hat{f}(k,m,n,a,t)e^{ir_{kmnat}s}e^{i<n,\theta>}\gamma_m(x)y_{\pm}^{ia-\frac{1}{2}}r^{it-1}dadt =$$

$$= \sum_{\pm}\sum_{(k,m,n)\in\mathbf{N}^{1+p+c}}\int_{\mathbb{R}^q}\int_{\mathbb{R}^{+c}}\hat{f}(k,m,n,a,t)r_{kmnrt}^{i\frac{1}{2}(|a|+|t|)}e^{ir_{kmnat}s}e^{i<n,\theta>}\gamma_m(\sqrt{r_{kmnrt}}x)y_{\pm}^{ia-\frac{1}{2}}r^{it-1}dadt.$$

Here, we used that the hyperbolic eigenfunctions are eigenfunctions of dilation operators.

It follows, formally, from the semi-classical normal form and from the eigenfunction expansion that

$$W_\gamma^{-1} T^{-1} \Delta T W_\gamma \sim \mathcal{L}^2 + f_o(I_1^e, ..., I_{2c}^{ch,Im}) + \frac{f_1(I_1^e, ..., I_{2c}^{ch,Im})}{\mathcal{L}} + \dots . \tag{4.3}$$

We now show that the intertwining operator is actually a standard Fourier Integral operator (in the Weyl operator, or isotropic, sense) and that (4.3) holds modulo the the kind of error stated in Theorem I.

The proof is again similar to the elliptic case, so we concentrate on the novel aspects and refer the reader to [Z.1, Proposition 3.4] for the remaining details. As before, we will not be as careful here as in [Z.1] to express things in weightless terms relative to metric rescalings.

(4.4) **Propostion** *$TW_\gamma T^{-1}$ is a (standard) Fourier integral operator, well-defined and invertible on the microlocal neighborhood (0.1) in $T^*(S^1_L \times \mathbb{R}^n)$.*

Sketch of Proof:

We first consider the unitarily equivalent operator $\tilde{T}\tilde{W}\tilde{T}^{-1}$ in the microlocal neighborhood (1.2.1) in the twisted model, with

$$\tilde{W} : \mathcal{H}_\alpha \to \mathcal{H}_\alpha \tag{4.5}$$

$$\tilde{W}(e^{ir_{kmnat}s} e^{i<n,\theta>} \gamma_m(x) y_\pm^{ia-\frac{1}{2}} \rho^{it-1}) := e^{ir_{kmnat}s} W_{r^{-1}_{kmnat}} e^{i<n,\theta>} \gamma_m(x) y_\pm^{ia-\frac{1}{2}} r^{it-1},$$

and with $\tilde{T}$ the dilation operator analogous to (4.2) but relative to the basis $e^{ir_{kmnat}s} e^{i<n,\theta>} \gamma_q(x) y_\pm^{ia-\frac{1}{2}} r^{it-1}$. We then factor $\tilde{T}\tilde{W}\tilde{T}^{-1}$ as the product $\tilde{T}\tilde{W}\tilde{T}^{-1} = j^*\tilde{T}V\tilde{T}^{-1}$ where:

$$V : \mathcal{H}_\alpha \to L^2_{loc}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n) \tag{4.6}$$

$$V = \Pi_{j=o}^\infty exp[iD_s^{-\frac{j}{2}} Q_{\frac{j}{2}}(s', y, D_y)]$$

that is,

$$Ve^{ir_{kmnat}s} e^{i<n,\theta>} \gamma_m(x) y_\pm^{ia-\frac{1}{2}} r^{it-1}) := e^{ir_{kmnat}s} W_{r^{-1}_{kmnat}}(s', x, y, w, D_x, D_y, D_w) e^{i<n,\theta>} \gamma_m(x) y_\pm^{ia-\frac{1}{2}} r^{it-1},$$

and where

$$j^* : C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n) \to C^\infty(\mathbb{R} \times \mathbb{R}^n) \tag{4.7}$$

$$j^* f(s, x) = f(s, s, x)$$

is the pullback under the partial diagonal embedding.

The discussion of V and the proof that j^*V is a standard Fourier Integral operators goes precisely as in [Z.1, Proposition 3.4]. The effect of the dilation is to convert the isotropic calculus into the pure polyhomogenous calculus (see also [G.1] for this aspect) and then the power of D_s insures that the phases are all homogeneous of degree 1 and vanishing to higher and higher order along γ (by one step as the index j increases by one unit). Hence the phase of the infinite product has only finitely many terms of a given vanishing order and converges as a formal power series in the transverse variable. A convergent product can be defined (by Borel summation) of the phase (cf. [Sj]).

The proposition then follows by expressing

$$TW_\gamma T^{-1} = T\mu(\mathcal{W})^* \tilde{T}^{-1} \tilde{T} W \tilde{T}^{-1} \tilde{T} \mu(\mathcal{W}) T^{-1}$$

and noting that $\tilde{T}\mu(\mathcal{W})T^{-1}$ is also a standard Fourier Integral operator. □

We now complete the proof of the quantum normal form Theorem I for $\sqrt{\Delta}$, stated in an equivalent form in terms of W_γ. As in the introduction, the notation $A \equiv B$ means that the complete (Weyl) symbol of $A - B$ vanishes to infinite order at γ and $O_j\Psi^m$ denotes the pseudodifferential operators of order m whose Weyl symbols vanish to order j at $(y, \eta) = (0, 0)$. Here, pseudodifferential operator can refer to either the standard polyhomogeneous kind, or to the mixed polyhomogeneous-isotropic kind as in $\Psi^k(S^1_L) \otimes \mathcal{W}^l$, in which case the total order is defined to be $m = k + l$. To simplify notation, we will denote the space of mixed operators of order m by $\Psi^m_{mx}(S^1_L \times \mathbb{R}^n)$.

(4.8) **Lemma** *Let $TW_\gamma T^{-1}$ be the Fourier Integral operator of Proposition (4.4), defined over a conic neighborhood of $R^+\gamma$ in $T^*(S^1_L \times \mathbb{R}^n)$. Then:*

$W_\gamma^{-1}T^{-1}\sqrt{\Delta}TW_\gamma \equiv P_1(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) + P_o(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) + \dots \quad mod \quad \oplus_{k=o}^{m+1} O_{2(m+1-k)}\Psi_{mx}^{1-k}(S^1_L \times \mathbb{R}^n),$

where

$$P_1(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) \equiv \mathcal{L} + \frac{p_1^{[2]}(I_1^e, ..., I_{2c}^{ch,Im})}{L\mathcal{L}} + \frac{p_2^{[3]}(I_1^e, ..., I_{2c}^{ch,Im})}{(L\mathcal{L})^2} + \dots \tag{4.9}$$

$$P_{-m}(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) \equiv \sum_{k=m}^{\infty} \frac{p_k^{[k-m]}(I_1^e, ..., I_{2c}^{ch,Im})}{(L\mathcal{L})^j}$$

with $p_k^{[k-m]}$, for m=-1,0,1,..., homogenous of degree l-m in the variables $(I_1^e, ..., I_{2c}^{ch,Im})$ and of weight -1.

Proof:

As a semi-classical expansion in the "parameter" $h = \frac{1}{L\mathcal{L}}$, (4.3) may be rewritten in the form :

$$W_\gamma^{-1}T^{-1}\sqrt{\Delta}TW_\gamma \sim \mathcal{L} + \frac{p_1(I_1^e, ..., I_{2c}^{ch,Im})}{L\mathcal{L}} + \frac{p_2(I_1^e, ..., I_{2c}^{ch,Im})}{(L\mathcal{L})^2} + \dots. \tag{4.10}$$

From the fact that the numerators $f_j(I_1^e, ..., I_{2c}^{ch,Im})$ in (4.3) are polynomials of degree j+2 and of weight -2, the numerators $p_k(I_1^e, ..., I_{2c}^{ch,Im})$ are easily seen to be polynomials of degree $k+1$ and of weight -1. Hence they may be expanded in homogeneous terms

$$p_k = p_k^{[k+1]} + p_k^{[k]} + \dots p_k^{[o]}, \tag{4.11}$$

with $p_k^{[j]}$ the term of degree j and still of weight -1. The right side of (4.12) can then be expressed as a sum of homogeneous operators:

$$P_1(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) + P_o(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) + \dots \tag{4.12}$$

with

$$P_1(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) \equiv \mathcal{L} + \frac{p_1^{[2]}(I_1^e, ..., I_{2c}^{ch,Im})}{L\mathcal{L}} + \frac{p_2^{[3]}(I_1^e, ..., I_{2c}^{ch,Im})}{(L\mathcal{L})^2} + \dots \tag{4.13}$$

$$P_{-m}(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) \equiv \sum_{k=m}^{\infty} \frac{p_k^{[k-m]}(I_1^e, ..., I_{2c}^{ch,Im})}{(L\mathcal{L})^k}.$$

We claim that:

$$W_\gamma^{-1}T^{-1}\sqrt{\Delta}TW_\gamma - [\mathcal{L} + \frac{p_1(I_1^e, ..., I_{2c}^{ch,Im})}{L\mathcal{L}} + \frac{p_2(I_1^e, ..., I_{2c}^{ch,Im})}{(L\mathcal{L})^2} + \dots + \frac{p_m(I_1^e, ..., I_{2c}^{ch,Im})}{(L\mathcal{L})^m}] \tag{4.14}$$

$$\in \oplus_{k=o}^{m+1} O_{2(m+1-k)}\Psi_{mx}^{1-k}(S^1_L \times \mathbb{R}^n).$$

Indeed, from the analysis of the remainder terms in the semi-classical normal form (see Lemma (3.1 (i)) and [Z.1, Lemma 2.22]), we have

$$P_1(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) - [\mathcal{L} + \frac{p_1^{[2]}(I_1^e, ..., I_{2c}^{ch,Im})}{L\mathcal{L}} + \dots + \frac{p_N^{[N+1]}(I_1^e, ..., I_{2c}^{ch,Im})}{(L\mathcal{L})^k}] \tag{4.15}$$

$$\in O_{2(N+2)}\Psi_{mx}^1(S^1_L \times \mathbb{R}^n)$$

and also

$$P_{-m}(\mathcal{L}, I_1^e, ..., I_{2c}^{ch,Im}) - \sum_{k=m}^{N} \frac{p_k^{[k-m]}(I_1^e, ..., I_{2c}^{ch,Im})}{(L\mathcal{L})^k} \in O_{2(N+1-m)}\Psi_{mx}^{-m}(S^1_L \times \mathbb{R}^n). \tag{4.16}$$

Hence the expansion (4.10) is also asymptotic in the sense of $\equiv$. For the statement of Theorem I in the introduction, it is only necessary to conjugate under $\mu(\tilde{W})$. The rest proceeds as in the elliptic case. □

5. Wave invariants and residue trace: Proof of Theorem B

The purpose of this section is to show that the wave invariants have precisely the same relation to the coefficients of the quantum normal form in the non-degenerate case that they have in the elliptic one. The characterization of the wave invariants in Theorem I will then follow from Theorem A of [Z.1].

We will need to use some further notation and results from [Z.1]: First, the kth wave invariant of a positive elliptic operator P at a non-degenerate closed bicharacteristic γ will be denoted $\tau_{\gamma k}(P)$. According to [Z.1, Proposition 4.2] we then have:

$$\tau_{\gamma k}(P) = \tau_{\gamma k}(P_1^{\leq 2k+4} + P_o^{\leq 2k+2} + \cdots + P^o_{-k-1}) \tag{5.1}$$

where $P_j^{\leq k}$ denotes the first k terms in the Taylor expansion of the jth homogeneous part of the complete symbol of P at γ. Thus, $\tau_{\gamma k}(P)$ involves the (2k+4)th jet of the principal symbol, the (2k+2)-jet of the subprincipal term, ..., up to the zero-jet of term of homogeneity order (-k-1).

As in [Z.1, §4], we will also rewrite the normal form in terms of of D_s and $H_{\alpha,\lambda,(\mu,\nu)}$ using that

$$\frac{p_\nu(I_1^e,\dots,I_{2c}^{ch,Im})}{(L\mathcal{R})^\nu} = \frac{p_\nu(I_1^e,\dots,I_{2c}^{ch,Im})}{(LD_s)^\nu}(I - \nu\frac{H_{\alpha,\lambda,(\mu,\nu)}}{LD_s} + \frac{1}{2}\nu(\nu-1)(\frac{H_{\alpha,\lambda,(\mu,\nu)}}{LD_s})^2 + \dots).$$

By (5.1), we can drop the $D_s^{-(k+1)+\nu}H^{k+1-\nu}_{\alpha,\lambda,(\mu,\nu)}$ and higher terms, so $\mathcal{D}_{k+1}$ can be written in the form

$$\mathcal{D}_{k+1} \equiv LD_s + H_{\alpha,\lambda,(\mu,\nu)} + \frac{\tilde{p}_1(I_1^e,\dots,I_{2c}^{ch,Im})}{LD_s} + \frac{\tilde{p}_2(I_1^e,\dots,I_{2c}^{ch,Im})}{(LD_s)^2} + \cdots + \frac{\tilde{p}_{k+1}(I_1^e,\dots,I_{2c}^{ch,Im})}{(LD_s)^{k+1}} \tag{5.2}$$

modulo terms which make no contribution to $\tau_{k\gamma}$.

We then use the fact ([Z.1, (4.3)],[Z.2]) that

$$\tau_{\gamma k}(P) = res D_t^k \psi_\epsilon(D_s, y, D_y)e^{itP}|_{t=L} \tag{5.3}$$

where res is the non-commutative residue and where $\psi_\epsilon(D_s, y, D_y)$ denotes a microlocal cut-off to the cone (1.2.1). Note that in contrast to the elliptic case, the microlocal cut-off cannot be constructed in $\mathcal{A}_{p,q,c}$ since the neighboorhoods given by $I < \epsilon\sigma$ in terms of mixed hyperbolic-elliptic actions are of infinite transverse symplectic volume. This does not pose a genuine problem, but accounts for a number of modifications to the elliptic case in [Z.1]. For the gauging elliptic operator we use LD_s.

To simplify the notation we will put

$$\mathcal{P}_{k+1}(D_s, I_1^e,\dots,I_{2c}^{ch,Im}) := \frac{\tilde{p}_1(I_1^e,\dots,I_{2c}^{ch,Im})}{LD_s} + \frac{\tilde{p}_2(I_1^e,\dots,I_{2c}^{ch,Im})}{(LD_s)^2} + \cdots + \frac{\tilde{p}_{k+1}(I_1^e,\dots,I_{2c}^{ch,Im})}{(LD_s)^{k+1}}. \tag{5.4}$$

so that:

$$\tau_{\gamma k}(\sqrt{\Delta}) = Res_{z=0} Tr D_t^k \psi_\epsilon(D_s, y, D_y) e^{it[\frac{1}{L}(2\pi LD_s + H_{\alpha,\lambda,(\mu,\nu)}) + \mathcal{P}_{k+1}]}(LD_s)^{-z}|_{t=L}. \tag{5.5}$$

As in the elliptic case, a key role will be played by the (formal) trace

$$T(\alpha,\lambda,(\mu,\nu)) := Tr e^{iH_{\alpha,\lambda,(\mu,\nu)}}.$$

Its precise definition is the following: Since $e^{iH_{\alpha,\lambda,(\mu,\nu)}}$ is an element of the metaplectic representation μ of $Mp(n,\mathbb{R})$, $T(\alpha,\lambda,(\mu,\nu))$ may be identified with the character Ch of μ evaluated at the associated element $P_\gamma = exp(\Xi_{H_{\alpha,\lambda,(\mu,\nu)}}) \in Mp(n,\mathbb{R})$. Here, $H_{\alpha,\lambda,(\mu,\nu)}$ denotes the quadratic function on $\mathbb{R}^{2n}$ which gives the complete Weyl symbol of the corresponding action operator, and as above $exp \circ \Xi$ denotes the flow at time 1 of its Hamilton vector field, or, more correctly, the lift to $Mp(n,\mathbb{R})$ which corresponds to $e^{iH_{\alpha,\lambda,(\mu,\nu)}}$ under μ.

Since $Mp(n,\mathbb{R})$ is a semi-simple Lie group, the character Ch is a real analytic function on the open dense subset $Mp(n,\mathbb{R})_{reg}$ of regular elements of $Mp(n,\mathbb{R})$, where it is given by the Harish-Chandra formula [Kn]. We will need below the explicit formula for $Ch(x)$ in terms of the eigenvalues of x. For elements of $Mp(n,\mathbb{R})$ not having 1 as an eigenvalue, we recall that $Ch(x)$ is given by

$$Ch(x) = \frac{i^\sigma}{\sqrt{|det(I-x)|}}$$

where σ is a certain Maslov index. For non-degenerate x with p pairs of eigenvalues $e^{\pm i\alpha_j}$ of modulus one, q pairs of positive real eigenvalues $e^{\pm\lambda_j}$ and c quadruplets of eigenvalues $e^{\pm(\mu_j \pm i\nu_j)}$, $Ch(x)$ is therefore given (up to a Maslov factor) by

$$T(\alpha, \lambda, (\mu, \nu)) = \Pi_{j=1}^{p} \frac{e^{\frac{1}{2}i\alpha_j}}{1 - e^{i\alpha_j}} \cdot \Pi_{j=1}^{q} \frac{e^{\frac{1}{2}\lambda_j}}{1 - e^{\lambda_j}} \cdot \Pi_{j=1}^{c} \frac{e^{\frac{1}{2}(\mu_j + i\nu_j)}}{1 - e^{\mu_j + i\nu_j}} \frac{e^{\frac{1}{2}(\mu_j - i\nu_j)}}{1 - e^{\mu_j - i\nu_j}}. \tag{5.6}$$

Here we have selected one eigenvalue ρ from each symplectic pair ρ, ρ^{-1} (see §1.1-2). The ambiguity is fixed by the Maslov factor i^{σ}, which can (and will) be ignored below for the sake of brevity.

We can now give:

Proof of Theorem B: Since $e^{2\pi i L D_s} \equiv I$ on $L^2(S^1_L)$ we have

$$a_{k\gamma} = \tau_{\gamma k}(\sqrt{\Delta}) =$$

$$Res_{z=0} Tr \psi_\epsilon(D_s, y, D_y)[\frac{1}{L} 2(\pi L D_s + H_{\alpha,\lambda,(\mu,\nu)}) + \mathcal{P}_{k+1}]^k e^{iH_{\alpha,\lambda,(\mu,\nu)}} e^{iL\mathcal{P}_{k+1}} (LD_s)^{-z}. \tag{5.7}$$

In view of the microlocal cutoff, the operator under the trace is of trace class for Rez sufficiently large. Indeed, in estimating the trace we may eliminate the unitary factors and we are then left with a pseudodifferential operator whose complete symbol is a polynomial in (σ, y, η) times a factor of $\sigma^{-Rez}\psi_\epsilon(\sigma, y, \eta)$. The integral in the transverse (y, η) variables is bounded by the volume of the ball $(y^2 + \eta^2) < \sigma$ and hence is of order σ^n. Since a pseudodifferential operator is Hilbert-Schmidt if its Weyl symbol is in L^2, the operator under the trace is Hilbert-Schmidt for $Rez > n$ and in particular is of trace class. Moreover, since it is the non-commutative residue of a Fourier Integral operator, one knows apriori that it admits a meromorphic continuation to $\mathbb{C}$ with at most simple poles [Z.2]. Hence the residue is well defined.

As in [Z.1], we view the trace as a function of the parameters $(\alpha, \lambda, \mu, \nu)$ and use the explicit form of the exponential in $e^{iH_{\alpha,\lambda,(\mu,\nu)}}$ to rewrite (5.7) in the form

$$Res_{z=0} \sum_{n=1}^{\infty} n^{-z} Tr \psi_\epsilon(D_s, y, D_y)\{[\frac{1}{L}(2\pi n + \sum_{j=1}^{p} \alpha_j D_{\alpha_j} + \sum_{j=1}^{q} \lambda_j \partial_{\lambda_j} +$$

$$+ \sum_{j=1}^{2c} (\mu_j \partial_{\mu_j} + \nu_j D_{\nu_j}) + \mathcal{P}_{k+1}(n, D_{\alpha_1}, \dots, D_{\nu_{2c}}, L)]^k e^{iL\mathcal{P}_{k+1}(n, D_{\alpha_1}, \dots, D_{\nu_{2c}}, L)} e^{iH_{\alpha,\lambda,(\mu,\nu)}}\}. \tag{5.8}$$

Here, we have used that D_s commutes with (y, D_y) to replace it by its eigenvalue in the s-trace, and we have repeatedly used identities of the form $F(D_x)e^{ixP} = F(P)e^{ixP}$ $(x \in \mathbb{R})$.

Since $\mathcal{P}_{k+1}(n, D_{\alpha_1}, \dots, D_{\nu_{2c}}, L)$ is a symbol of order -1 in n with coefficients given by polynomials in the operators D_{α_j} (etc.), we can expand the kth power in (5.8) as an operator-valued polyhomogeneous function of n. At least formally, we can also expand the exponential $e^{iL\mathcal{P}_{k+1}(n, D_{\alpha_1}, \dots, D_{\nu_{2c}}, L)}$ in a power series and then expand each term in the power series as a polynomial in n^{-1}. Collecting powers of n, the right side of (5.8) may be written in the form

$$Res_{z=0} \sum_{n=1}^{\infty} \sum_{j=o}^{\infty} n^{-z+k-j} \mathcal{F}_{k,k-j}(D_{\alpha_1}, \dots, D_{\mu_c + i\nu_c}, D_{\mu_c - i\nu_c}) Tr \psi_\epsilon(n, y, D_y) e^{iH_{\alpha,\lambda,(\mu,\nu)}}, \tag{5.9}$$

with $\mathcal{F}_{k,k-j}(D_\alpha, \dots, D_{\mu_c + i\nu_c}, D_{\mu_c - i\nu_c} L)$ the coefficient of n^{k-j} in (5.8). The expansion of the exponential is justified as in the elliptic case: as in [Z.1, (4.29)] we may write

$$e^{iL\mathcal{P}_{k+1}} := e_N(iL\mathcal{P}_{k+1}) + (iL\mathcal{P}_{k+1})^{N+1} b_N(iL\mathcal{P}_{k+1})$$

with $e_N(ix) = 1 + ix + \dots + \frac{(ix)^N}{N!}$, with $\mathcal{P}_{k+1}$ short for $\mathcal{P}_{k+1}(n, D_{\alpha_1}, \dots, D_{\nu_{2c}})$ and with $b_N(ix)$ a bounded function. The e_N term contributes a finite number of terms of the desired form (5.9). For the remainder, we expand $(iL\mathcal{P}_{k+1})^{N+1}$ as a polynomial in n^{-1} with coefficients given by operators $Q_{Np}(D_{\alpha_1}, \dots, D_{\nu_{2c}})$ and observe that each term has a factor of n^{-N-1}. For each such term, we remove the coefficient operator Q_{Np} from the sum $\sum_n$, as above, leaving only the factor of b_N. Since $b_N(ix)$ is a bounded function, it follows that $b_N(iL\mathcal{P}_{k+1})$ is a bounded operator on L^2; and since each term of the resulting sum has at least the

factor $n^{-z-N-1+k}$ (possibly multiplied by a further negative power of n), we see that the remainder is a sum of terms of the form

$$Res_{z=0}Q_{Np}(D_{\alpha_1},\dots,D_{\nu_{2c}})\sum_{kn} n^{-z+k-N-1-l}b_N(i\mathcal{P}_{k+1}(n,D_{\alpha_1},\dots))Tr\psi_\epsilon(n,y,D_y)e^{iH_{\alpha,\lambda,(\mu,\nu)}}. \tag{5.10}$$

We then observe that the sum is bounded by $\sum_{m=1}^\infty m^{-Rez-N-1+k+n}$, hence converges absolutely and uniformly for $Rez > -N+k+n$. It follows that for $N>(n+k)$ the sum in (5.10) defines a holomorphic function of z in a half-plane containing $z=0$ and since the operations of taking the residue in z and derivatives in α commute, each term (5.10) is zero. This justifies (5.10) and shows that it is actually a finite sum in j, say $j<M$ (in fact M=(k+1)(n+k+1)).

The residue in (5.10) is therefore well-defined and independent of ϵ. Since $Tr\psi_\epsilon(n,y,D_y)e^{iH_{\alpha,\lambda,(\mu,\nu)}}\to T(\alpha,\lambda,\mu,\nu)$ in the sense of distributions as $\epsilon\to\infty$ we must have

$$\begin{aligned} a_{\gamma k} &= Res_{z=0}\sum_{m=1}^\infty\sum_{j=o}^M m^{-z+k-j}\mathcal{F}_{k,k-j}(D_\alpha,\dots,D_{\mu_c+i\nu_c},D_{\mu_c-i\nu_c})\ \ T(\alpha,\lambda,\mu,\nu) \\ &= Res_{z=0}\sum_{j=0}^M\zeta(z+j-k)\mathcal{F}_{k,k-j}(D_\alpha,\dots,D_{\mu_c+i\nu_c},D_{\mu_c-i\nu_c})\ \ T(\alpha,\lambda,\mu,\nu). \end{aligned} \tag{5.11}$$

Here, ζ is the Riemann zeta-function, which has only a simple pole at $s=1$ with reside equal to one. It follows that the only term contributing to (5.11) is that with $j=k+1$ and hence we have

$$a_{\gamma k}=\mathcal{F}_{k,-1}(D_\alpha,\dots,D_{\mu_c+i\nu_c},D_{\mu_c-i\nu_c})T(\alpha,\lambda,\mu,\nu). \tag{5.12}$$

It follows that the wave invariants consist of the geometric data contained in the coefficients of the $\mathcal{F}_{k,-1}$'s, and hence in the normal form coefficients. But the algorithm for constructing the normal form is essentially the same as in the elliptic case and so the geometric data entering into the normal form coefficients is of precisely the same kind. □

6. Inverse Problems: Proofs of Theorem II and Corollary II.1

Our first goal in this section is to prove that the wave invariants of $\gamma,\gamma^2,\dots$ determine the quantum normal form coefficients at γ.

Proof of Theorem II:

We recall that the quantum normal coefficients are the coefficients of the action monomials in the action polynomials $\tilde{p}_j(I_1^e,\dots,I_{2c}^{ch,Im})$ of (5.2). These coefficients determine, and are determined by, the coefficients of the monomials in the polynomials $p_\nu(I_1^e,\dots,I_{2c}^{ch,Im})$ in Theorem B. They also corresponding bi-uniquely to the coefficients of the classical action monomials in the complete symbols of either set of action polynomials.

We also observe that the quantum normal form coefficients determine, and are determined by, the coefficients of the constant coefficient partial differential operator (PDO)

$$\mathcal{F}_{k,-1}(D_\alpha,D_\lambda,D_{\mu+i\nu},D_{\mu-i\nu}) := \sum_{(a,b,c_1,c_2)\in\mathbb{N}^n:|a|+|b|+|c_1|+|c_2|\le k+1} C_{k;abc_1c_2}D_\alpha^aD_\lambda^bD_{\mu+i\nu}^{c_1}D_{\mu-i\nu}^{c_2} \tag{6.1k}$$

where $a\in\mathbb{N}^p, b\in\mathbb{N}^q, c_1,c_2\in\mathbb{N}^c$. This can be proved easily by induction on k: In the case k=1, $\mathcal{F}_{k,-1}$ is obtained from $\tilde{p}_1$ by substituting the variables D_{α_1}, etc. in for the variables I_1^e, etc. Assuming inductively that we have determined the coefficients of $\tilde{p}_1,\dots,\tilde{p}_k$ from those of $\mathcal{F}_{1,-1},\dots,\mathcal{F}_{1,k}$, we note that $\tilde{p}_{k+1}$ contributes to the residue (5.7) for the first time at the $k+1$st stage. Since it comes with the denominator D_s^{k+1}, it only contributes to the residue when composed with D_s^k. From the form of (5.7) it is clear that only one term involving $\tilde{p}_{k+1}$ contributes non-trivially, and that is the one which appears in the linear term in the expansion of the exponential. Hence, its contribution to $\mathcal{F}_{k+1,-1}$ is again just the substitution of the variables D_{α_1}, etc. in for the variables I_1^e, etc.

We next observe that the quantum normal form of Δ at any iterate γ^N of γ is the same as for the primitive γ itself. Hence the PDO $\mathcal{F}_{k,-1}$ is independent of the number N of iterations. On the other hand,

under the iteration $\gamma \to \gamma^N$, the Poincare map transforms by $P_{\gamma^N} \to P_\gamma^N$. Therefore the expression in (5.12) for the kth wave invariant of γ^N is given by:

$$\mathcal{F}_{k,-1}(D_{\alpha'}, D_{\lambda'}, D_{\mu'+i\nu'}, D_{\mu'-i\nu'})\cdot$$

$$(6.2k,N) \qquad \cdot\Pi_{j=1}^p \frac{e^{\frac{1}{2}i\alpha'_j}}{(1-e^{i\alpha'_j})} \cdot \Pi_{j=1}^q \frac{e^{\frac{1}{2}\lambda'_j}}{(1-e^{\lambda'_j})} \cdot \Pi_{j=1}^c \frac{e^{\frac{1}{2}(\mu'_j+i\nu'_j)}}{(1-e^{(\mu'_j+i\nu'_j)})} \frac{e^{\frac{1}{2}(\mu'_j-i\nu'_j)}}{(1-e^{(\mu'_j-i\nu'_j)})}|_{(\alpha',\lambda',\mu',\nu')=N(\alpha,\lambda,\mu,\nu)}.$$

It therefore suffices to prove that for all k the coefficients of the PDO $\mathcal{F}_{k,-1}$ can be determined from its values (6.2 k, N) on $T(N\alpha, N\lambda, N\mu, N\nu)$ for $N = \pm 1, \pm 2, \ldots$.

We begin the proof by noting that (6.2 k, N) can be rewritten as:

$$(6.3k,N) \qquad \Pi_{j=1}^p e^{\frac{1}{2}iN\alpha_j} \cdot \Pi_{j=1}^q e^{\frac{1}{2}N\lambda_j} \cdot \Pi_{j=1}^c e^{\frac{1}{2}N(\mu_j+i\nu_j)} e^{\frac{1}{2}N(\mu_j-i\nu_j)}.$$

$$\cdot\mathcal{F}_{k,-1}(D_{\alpha'}+\frac{1}{2}, D_{\lambda'}+\frac{1}{2}, D_{\mu'+i\nu'}+\frac{1}{2}, D_{\mu'-i\nu'}+\frac{1}{2})[\Pi_{j=1}^p \frac{1}{(1-e^{i\alpha'_j})}$$

$$\cdot\Pi_{j=1}^q \frac{1}{(1-e^{\lambda'_j})} \cdot \Pi_{j=1}^c \frac{1}{(1-e^{(\mu'_j+i\nu'_j)})} \frac{1}{(1-e^{(\mu'_j-i\nu'_j)})}]|_{(\alpha',\lambda',\mu',\nu')=N(\alpha,\lambda,\mu,\nu)}.$$

Making the substitutions $D_\alpha \to D_\alpha + \frac{1}{2}$ (etc.) in (6.1 k) we obtain a new PDO whose coefficients $C'_{k;abc_1c_2}$ correspond in a bi-unique way with the original $C_{k;abc_1c_2}$'s. Hence it will suffice to show that we can determine the $C_{k;abc_1c_2}$'s from the values (6.3 k,N).

To do so, we will regard (6.3 k, N) as the values at integral points $z = N$ of a function of z. From the fact that

$$D_\alpha(1-e^{i\alpha})^{-1} = [(1-e^{i\alpha})^{-2} - (1-e^{i\alpha})^{-1}]$$

we see that this function is a polynomial in $(1-e^{iz\alpha_j})^{-1}, (1-e^{z\lambda_j})^{-1}, (1-e^{z(\mu+i\nu)})^{-1}, (1-e^{z(\mu-i\nu)})^{-1}$. We clear the denominators to obtain the entire function

$$(6.4k,z) \qquad \sum_{(a,b,c_1,c_2)\in\mathbb{N}^n : |a|+|b|+|c_1|+|c_2|\le k+1} C'_{k;abc_1c_2}[\Pi_{j=1}^p(1-e^{iz\alpha_j}) \cdot \Pi_{j=1}^q\cdot$$

$$\cdot(1-e^{z\lambda_j}) \cdot \Pi_{j=1}^c(1-e^{z(\mu_j+i\nu_j)})(1-e^{z(\mu_j-i\nu_j)})]^{(k+1)} \cdot \Pi_{j=1}^p e^{\frac{1}{2}iz\alpha_j} \cdot \Pi_{j=1}^q e^{\frac{1}{2}z\lambda_j} \cdot \Pi_{j=1}^c e^{\frac{1}{2}z(\mu_j+i\nu_j)} e^{\frac{1}{2}z(\mu_j-i\nu_j)}.$$

$$\cdot D_{\alpha'}^a D_{\lambda'}^b D_{\mu'+i\nu'}^{c_1} D_{\mu'-i\nu'}^{c_2} \cdot\cdot [\Pi_{j=1}^p \frac{1}{(1-e^{i\alpha'_j})} \cdot\cdot \Pi_{j=1}^q \frac{1}{(1-e^{\lambda'_j})} \cdot \Pi_{j=1}^c \frac{1}{(1-e^{(\mu'_j+i\nu'_j)})} \frac{1}{(1-e^{(\mu'_j-i\nu'_j)})}]|_{(\alpha',\lambda',\mu',\nu')=z(\alpha,\lambda,\mu,\nu)}$$

which is an exponential polynomial of the form

$$(6.5k) \qquad \sum_{\beta\in\mathbb{N}^n} c_{k;\beta} e^{z\langle \beta + (\frac{1}{2},\ldots,\frac{1}{2}), (\alpha,\lambda\mu,\nu)\rangle}.$$

(6.6) Lemma 1 *The coefficients $C_{k;abc_1c_2}$ can be determined from the coefficients $c_{k;\beta}$ in (6.5 k).*

Proof: We first show that the coefficients $C_{k;abc_1c_2}$ with $|a|+|b|+|c_1|+|c_2| = k+1$ can be determined from the $c_{k;\beta}$'s. Indeed, before multiplying by

$$[\Pi_{j=1}^p(1-e^{iz\alpha_j}) \cdot \Pi_{j=1}^q(1-e^{z\lambda_j}) \cdot \Pi_{j=1}^c(1-e^{z(\mu_j+i\nu_j)})(1-e^{z(\mu_j-i\nu_j)})]^{(k+1)}.$$

$$\cdot\Pi_{j=1}^p e^{\frac{1}{2}iz\alpha_j} \cdot \Pi_{j=1}^q e^{\frac{1}{2}z\lambda_j} \cdot \Pi_{j=1}^c e^{\frac{1}{2}z(\mu_j+i\nu_j)} e^{\frac{1}{2}z(\mu_j-i\nu_j)}$$

$C'_{k;abc_1c_2}$ is uniquely determined as the coefficient of the monomial

$$\Pi_{j=1}^p(1-e^{iz\alpha_j})^{-(a_j+1)} \cdot\cdot \Pi_{j=1}^q \cdot (1-e^{z\lambda_j})^{-(b_j+1)} \cdot \Pi_{j=1}(1-e^{z(\mu_j+i\nu_j)})^{-(c_{1j}+1)}(1-e^{z(\mu_j-i\nu_j)})^{-(c_{2j}+1)}.$$

Since we are multiplying thru by a quantity independent of a, b, c_1, c_2, it follows that $C'_{k;abc_1c_2}$ is uniquely determined as the coefficient the of the monomial

$$\Pi_{j=1}^p(1-e^{iz\alpha_j})^{(k+1)-(a_j+1)} e^{\frac{1}{2}iz\alpha_j} \cdot \Pi_{j=1}^q(1-e^{z\lambda_j})^{(k+1)-(b_j+1)} e^{\frac{1}{2}z\lambda_j}.$$

$$\cdot\Pi_{j=1}^c(1-e^{z(\mu_j+i\nu_j)})^{(k+1)-(c_{1j}+1)}(1-e^{z(\mu_j-i\nu_j)})^{(k+1)-(c_{2j}+1)} e^{\frac{1}{2}z(\mu_j+i\nu_j)} e^{\frac{1}{2}z(\mu_j-i\nu_j)}.$$

Expanding into an exponential polynomial, we find that $C'_{k;abc_1c_2}$ is uniquely determined as the coefficient of the monomial

$$\Pi_{j=1}^{p} e^{iz((k+1)-(a_j+1)+\frac{1}{2})\alpha_j} \cdot \Pi_{j=1}^{q} e^{z((k+1)-(b_j+1)+\frac{1}{2})\lambda_j} \cdot$$
$$\cdot \Pi_{j=1}^{c} e^{z((k+1)-(c_{1j}+1)+\frac{1}{2})(\mu_j+i\nu_j)} e^{z((k+1)-(c_{2j})+\frac{1}{2})(\mu_j-i\nu_j)}).$$

Uniqueness follows from the fact that the vector $\beta+\frac{1}{2}$ is a minimal element of the set of exponent vectors occuring in (6.5 k). Since $C'_{k;abc_1c_2} = C_{k;abc_1c_2}$ when $|a|+|b|+|c_1|+|c_2| = k+1$, we have determined $C_{k;abc_1c_2}$.

We then remove the $C_{k;abc_1c_2}(D_\alpha+\frac{1}{2})^a(D_\lambda+\frac{1}{2})^b(D_{\mu+i\nu}+\frac{1}{2})^{c_1}(D_{\mu-i\nu}+\frac{1}{2})^{c_2}$ terms with $|a|+|b|+|c_1|+|c_2| = k+1$ in (6.5 k). This leaves only terms with coefficients $C_{k;abc_1c_2}$ with $|a|+|b|+|c_1|+|c_2| \le k$. Hence we can continue the process of recovering coefficients until the end. $\square$

Let us now rewrite

$$\sum_{\beta\in\mathbb{N}^n} c_{k;\beta} e^{z\langle\beta+(\frac{1}{2},\ldots,\frac{1}{2}),(\alpha,\lambda,\mu,\nu)\rangle} \tag{6.6a}$$

in the form

$$\sum_{j=1}^{M} a_{jk} e^{z\omega_j}. \tag{6.6b}$$

(6.7) Lemma *The complex exponents ω_j in (6.6b), together with π, are independent over the rationals. Moreover, the coefficients $c_{k:\beta}$ can be determined from the coefficients a_{jk}*

Proof: The ω_j's are rational linear combinations of the exponents $\alpha_j, \lambda_j, \mu_j, \nu_j$, which by assumption are independent, with π, over the rationals. This independence also implies that the exponents $\langle\beta+(\frac{1}{2},\ldots,\frac{1}{2}),(\alpha,\lambda,\mu,\nu)\rangle$ are all distinct. Hence the coefficients in (6.6a)-(6.6b) are the same. $\square$

The proof of Theorem II is thus reduced to the following general statement about exponential polynomials.

(6.8) Lemma *Suppose that the exponents ω_j of an exponential polynomial (6.6b) are independent (with π) over the rationals. Then the coefficients a_{jk} of can be determined from the values of this polynomial at $z = N \in \mathbb{N}$.*

Proof: If not, there would exist a polynomial with the given complex frequencies which vanished at all integers $z = N$. But the different terms $e^{z\omega_j}, e^{z\omega_k}$ have different exponential growth rates along $z = N$ or $z = -N$ ($N \in \mathbb{N}$) unless $Re\omega_j = Re\omega_k$. Let us write the large sum as a sum of smaller sums with a common $Re\omega$. Each of the smaller sums must separately vanish for $z \in \mathbb{N}$. Multiply each one by the relevant factor of $e^{-Re\omega}$. Each then turns into an exponential polynomial with imaginary exponents, which vanishes for all $z = N$. Since the frequencies are independent(with π) over $\mathbf{Q}$, each of these polynomials must vanish identically if it vanishes at integral points. Hence the coefficients a_{jk} are uniquely determined by values of the large sum at integral points. $\square$

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